

GEOMETRIC DEFORMATIONS OF ORTHOGONAL AND SYMPLECTIC GALOIS REPRESENTATIONS

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ABSTRACT. For a representation over a finite field of characteristic p of the absolute Galois group of the rationals, we study the existence of a lift to characteristic zero that is geometric in the sense of the Fontaine-Mazur conjecture. For two-dimensional representations, Ramakrishna proved that under technical assumptions odd representations admit geometric lifts. We generalize this to higher dimensional orthogonal and symplectic representations. The key innovation is the definition and study of a deformation condition at primes where the representation is ramified generalizing the minimally ramified deformation introduced for GL_n by Clozel, Harris, and Taylor. This requires an understanding of nilpotent orbits and centralizers of nilpotent elements in the relative situation, not just over fields.

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1. INTRODUCTION

Before the proof by Khare and Winterberger [KW09a] [KW09b] that irreducible odd representations

$$\bar{\rho} : \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \rightarrow \mathrm{GL}_2(\bar{\mathbf{F}}_p)$$

are modular, the lifting result of [Ram02] together with the Fontaine-Mazur conjecture provided evidence for Serre's conjecture. Ramakrishna's result shows that under technical hypotheses all odd residual representations admit lifts to characteristic zero that are geometric in the sense of the Fontaine-Mazur conjecture. Assuming that conjecture, the resulting lifts would be modular as predicted by Serre's conjecture. Generalizations of Serre's conjecture to groups other than GL_2 have been proposed, most recently by Gee, Herzig, and Savitt [GHS15], which naturally leads to the problem of producing geometric lifts of Galois representations for groups other than GL_2 .

Let K be a finite extension of \mathbf{Q} with absolute Galois group Γ_K . Suppose k is a finite field of characteristic p , \mathcal{O} the ring of integers in a p -adic field with residue field k , and G is a reductive group defined over \mathcal{O} . For a continuous representation $\bar{\rho} : \Gamma_K \rightarrow G(k)$, in light of these conjectures

it is important to study when there exists a continuous representation $\rho : \Gamma_K \rightarrow G(\mathcal{O})$ lifting $\bar{\rho}$ that is geometric (using $G \hookrightarrow \mathrm{GL}_n$).

When $G = \mathrm{GSp}_m$ or $G = \mathrm{GO}_m$, we produce geometric lifts in favorable conditions. The exact hypotheses needed are somewhat complicated. We will state a simple version now, and defer a more detailed statement to Theorem 3.12. It is essentially that $\bar{\rho}$ is odd (as discussed in Remark 1.2, forcing K to be totally real) and that $\bar{\rho}$ restricted to the decomposition group at p “looks like the reduction of a crystalline representation with *distinct* Hodge-Tate weights”. More precisely, we assume p is unramified in K and that at places v above p , the representation $\bar{\rho}|_{\Gamma_{K_v}}$ is torsion crystalline with Hodge-Tate weights in an interval of length $\frac{p-2}{2}$, so it is Fontaine-Laffaille. It is crucial that for each \mathbf{Z}_p -embedding of \mathcal{O}_{K_v} in $\overline{\mathcal{O}_{K_v}}$, the Fontaine-Laffaille weights for $\bar{\rho}|_{\Gamma_{K_v}}$ with respect to that embedding are pairwise distinct (these notions will be reviewed in §8).

For Ramakrishna’s method to apply, it is also essential that $G'(k) \subset \bar{\rho}(\Gamma_K)$ where G' is the derived group. Ramakrishna’s method requires certain technical conditions which follow from this assumption on the image provided that $p > \max(17, 2(m-1))$ (this restriction on p is not optimized: see Remark 3.9.) Let $\mu : G \rightarrow \mathbf{G}_m$ be the similitude character, and define $\bar{\nu} = \mu \circ \bar{\rho} : \Gamma_K \rightarrow k^\times$. Suppose there is a lift $\nu : \Gamma_K \rightarrow W(k)^\times$ that is Fontaine-Laffaille at all places above p .

Theorem 1.1. *Under these assumptions, there exists a geometric lift $\rho : \Gamma_K \rightarrow G(\mathcal{O})$ of $\bar{\rho}$ where \mathcal{O} is the ring of integers in a finite extension of \mathbf{Q}_p with residue field containing k such that $\mu \circ \rho = \nu$. More precisely, ρ is ramified at finitely many places of K , and for every place v of K above p the representation $\rho|_{\Gamma_{K_v}}$ is Fontaine-Laffaille and hence crystalline.*

This provides evidence for generalizations of Serre’s conjecture. In contrast, when $G = \mathrm{GL}_n$ with $n > 2$, the representation $\bar{\rho}$ cannot be odd, and the method does not apply. In such cases, there is no expectation that such lifts exist.

To produce lifts, we use a generalization of Ramakrishna’s method also used in [Pat15]. It works by establishing a local-to-global result for lifting Galois representations subject to local constraints (Proposition 2.10). Let ρ be a lift of $\bar{\rho}$ to $\mathcal{O}/\mathfrak{m}^n$ where \mathfrak{m} is the maximal ideal of \mathcal{O} . Provided a cohomological obstruction vanishes, it is possible to lift ρ to $\mathcal{O}/\mathfrak{m}^{n+1}$ subject to local constraints if (and only if) it is possible to lift $\rho|_{\Gamma_v}$ to $\mathcal{O}/\mathfrak{m}^{n+1}$ for all v in a fixed set of places of K containing the places above p and the places where $\bar{\rho}$ is ramified. Allowing controlled ramification at additional primes kills this obstruction for odd representations.

It remains to pick local deformation conditions above p and at places where $\bar{\rho}$ is ramified which are liftable in the sense that it is always possible to suitably lift $\rho|_{\Gamma_v}$. At p , we define a *Fontaine-Laffaille deformation condition* in §9 by using deformations arising from Fontaine-Laffaille modules that carry extra data corresponding to a symmetric or alternating pairing.

At a prime $\ell \neq p$ where $\bar{\rho}$ is ramified, we generalize the minimally ramified deformation condition defined for GL_n in [CHT08, §2.4.4]. In simple cases, this deformation condition controls the ramification of ρ by controlling deformations of a unipotent element \bar{g} of $\mathrm{GL}_n(k)$. There is a natural parabolic k -subgroup containing \bar{g} , and the deformation condition is analyzed by deforming this parabolic subgroup and then lifting \bar{g} inside this subgroup. This idea does *not* work for other algebraic groups. In Example 1.5 and §6.4, we discuss an explicit example in GSp_4 where the analogous deformation based on parabolics is provably not liftable. In §6 and §7, we define a *minimally ramified deformation condition* by instead requiring that \bar{g} deform so that “it lies in the same unipotent orbit as \bar{g} ,” and explain that the problems with the deformation condition based on parabolics are a general phenomenon. The discovery and study of this deformation condition at ramified places $\ell \neq p$ is the main contribution of this work. For GL_n , our notion agrees with minimally ramified deformation of [CHT08], but for other groups it is a genuinely different, liftable deformation condition.

In the remainder of the introduction, we discuss some additional background and give a more detailed overview of the proof.

1.1. Serre’s Conjecture and Geometric Lifts. We are interested in generalizations of Ramakrishna’s lifting result to split reductive groups beyond GL_2 , in particular symplectic and orthogonal groups. Generalizations of Serre’s conjecture have been proposed in this setting, and most of the effort has been to find the correct generalization of the oddness condition and the weight (see for example the discussion in [GHS15], especially §2.1). The general flavor of these generalizations is that an odd irreducible Galois representation will be automorphic in the sense that it appears in the cohomology of an $\overline{\mathbf{F}}_p$ -local system on a Shimura variety. For a general split reductive group, there is no expectation that such representations will lift to characteristic zero. For example, as discussed in [CHT08, §1] the Taylor-Wiles method would work only if

$$(1.1) \quad [K : \mathbf{Q}] (\dim G - \dim B) = \sum_{v|\infty} \dim H^0(\mathrm{Gal}(\overline{K}_v/K_v), \mathrm{ad}^0(\overline{\rho}))$$

where B is a Borel subgroup of G and $\mathrm{ad}^0(\overline{\rho})$ is the adjoint representation of Γ_K on the Lie algebra of the derived group of G . Only under such a “numerical coincidence” do we expect to obtain automorphy lifting theorems, and hence expect geometric lifts. This coincidence cannot hold for GL_n when $n > 2$, but can hold for $G = \mathrm{GSp}_{2n}$ and $G = \mathrm{GO}_m$, and for the group \mathcal{G}_n related to GL_n considered in [CHT08]. This coincidence is also essential to generalizing Ramakrishna’s method.

Remark 1.2. Following [Gro], we say that $\overline{\rho} : \Gamma_K \rightarrow G(k)$ is *odd* if for each archimedean place v and complex conjugation $c_v \in \Gamma_v$ (well-defined up to conjugacy), $\mathrm{ad}(\overline{\rho}(c_v))$ is a split Cartan involution for $\mathfrak{g}' := \mathrm{Lie} G^{\mathrm{ad}}$. Recall that for any involution τ of \mathfrak{g}' ,

$$\dim (\mathfrak{g}')^\tau \geq \dim G - \dim B$$

A *split Cartan involution* is an involution for which this is an equality. If K is totally real and $\overline{\rho}$ is odd, (1.1) holds. There are odd representations for symplectic and orthogonal groups, but no odd representations for GL_n when $n > 2$ (for more details, see [Pat15, §4.5]). These are cases in which we expect geometric lifts, and where Ramakrishna’s method generalizes.

There is a less restrictive notion of oddness introduced in [BV13, §6], and the automorphy lifting theorems in [CG12] apply beyond the regime where (1.1) holds.

Ramakrishna developed his lifting technique when $K = \mathbf{Q}$ and $G = \mathrm{GL}_2$ in [Ram99] and [Ram02], and produced geometric lifts. There have been various reformulations and generalizations that our results build on. In particular, the formalism developed in [Tay03] (still in the case of GL_2) suggested that it should be possible to generalize the technique to algebraic groups beyonds GL_2 . Attempts were made in [Ham08] and [Man09] to generalize the technique to GL_n , but ran into the obstruction that there were no odd representations for $n > 2$. The results in [Ham08] simply assume the existence of liftable local deformation conditions which probably do not exist, but do provide a nice model for generalizing Ramakrishna’s method. In contrast, [Man09] constructs local deformation conditions but does not aim to produce geometric lifts.

For groups beyond GL_n , [CHT08] gave a lifting result for a group \mathcal{G}_n related to GL_n which admits odd representations. Studying the local deformation conditions for \mathcal{G}_n reduced to studying representations valued in GL_n . At primes above p , [CHT08] studied a deformation condition based on Fontaine-Laffaille theory which is generalized in §9. The idea of doing so goes back to [Ram93]. (They also discussed a deformation condition based on the notion of ordinary representations which is not used in their lifting result). At primes not above p but where $\overline{\rho}$ is ramified, they defined a *minimally ramified* deformation condition, which we generalize in §6 and §7; this generalization is non-obvious and is our main innovation.

Building on this, Patrikis’ unpublished undergraduate thesis [Pat06] explored Ramakrishna’s method for symplectic groups. In particular, it generalized Ramakrishna’s method to the group GSp_n , and generalized the Fontaine-Laffaille deformation condition at p . It did not generalize the minimally ramified deformation condition, so can only be applied to residual representations

$\Gamma_{\mathbf{Q}} \rightarrow \mathrm{GSp}_n(k)$ which are unramified away from p , a stringent condition. Our results at p in §9 are a generalization of Patrikis' study of the Fontaine-Laffaille deformation condition.

More recently, Patrikis used Ramakrishna's method to produce geometric representations with exceptional monodromy [Pat15]. This involves generalizing Ramakrishna's method to any connected reductive group G and then modifying the technique to deform a representation valued in the principal $\mathrm{SL}_2 \subset G$ (coming from a modular form) to produce a geometric lift with Zariski-dense image. The generalization of Ramakrishna's method to apply to reductive groups is independently carried out in the author's thesis with only minor technical differences, so in §3 we refer the reader to [Pat15] for proofs. Our extensive study of local deformation conditions is not needed in [Pat15]: as the goal there is just to produce examples of geometric representations with exceptional monodromy, he could avoid generalizing the minimally ramified deformation condition.

Remark 1.3. There is also a completely different technique to produce lifts based on automorphy lifting theorems. For example, Khare and Winterberger use it in their proof of Serre's conjecture: see [KW09b, §4] especially the proof of Corollary 4.7. The finite generation needed in that argument comes from relating the Galois deformation ring to a Hecke algebra. It should be possible to deduce Theorem 1.1 from potential automorphy theorems.

1.2. Generalizing Ramakrishna's Method. We now outline the generalization of Ramakrishna's method. This part of the argument, with only minor technical variation, has also been carried out in [Pat15], and provides a framework for producing lifts if we can construct appropriate local deformation conditions. Fix a prime p and finite field k of characteristic p . Let S be a finite set of places of a number field containing the places above p and the archimedean places, and define Γ_S to be the Galois group of the maximal extension of K unramified outside of S . Consider a continuous representation $\bar{\rho} : \Gamma_S \rightarrow G(k)$ where G is a smooth affine group scheme over the ring of integers \mathcal{O} in a p -adic field such that the identity components of the fibers are reductive. We are mainly interested in the case that $G = \mathrm{GSp}_m$ or $G = \mathrm{GO}_m$; the latter may have disconnected fibers. (In the relative setting, by definition reductive groups have connected fibers, so we must work in slightly greater generality as discussed as the start of §2.1.)

We assume that p is very good for G (Definition 2.2), so the Lie algebra of the derived group of G° is a direct summand of the Lie algebra of G : we denote this summand with adjoint action of Γ_K by $\mathrm{ad}^0(\bar{\rho})$. The cohomology of this Galois module controls the deformation theory of $\bar{\rho}$. The hope would be to use deformation theory to produce $\rho_n : \Gamma_S \rightarrow G(\mathcal{O}/\mathfrak{m}^n)$ such that $\rho_1 = \bar{\rho}$, ρ_n lifts ρ_{n-1} for $n \geq 2$, and such that ρ_n satisfies a deformation condition at places above p for which the inverse limit

$$\rho = \varprojlim \rho_n : \Gamma_S \rightarrow G(\mathcal{O})$$

restricted to the decomposition group Γ_v would be a lattice in a de Rham (or crystalline) representation for places v of K above p . This inverse limit would then be the desired geometric lift of $\bar{\rho}$. Only after a careful choice of local deformation conditions and enlarging the set S will this work. Furthermore, defining these deformation conditions may require making an extension of k , which is harmless for our applications and is why we only require that the residue field of \mathcal{O} contains k .

Proposition 2.10 shows that a lift exists subject to a global deformation condition \mathcal{D}_S provided the dual Selmer group $H_{\mathcal{D}_S^\perp}^1(\Gamma_S, \mathrm{ad}^0(\bar{\rho})^*)$ vanishes. This Galois cohomology group is defined in (2.1), and encodes information about all of the local deformation conditions imposed. When it vanishes, there exists a lift of ρ_n to ρ_{n+1} satisfying local deformation conditions for $v \in S$ provided there exist lifts of $(\rho_n)|_{\Gamma_v}$ satisfying the deformation condition for all $v \in S$. This can be expressed as a local-to-global principle for lifting Galois representations with an obstruction lying in the cohomology group $H_{\mathcal{D}_S^\perp}^1(\Gamma_S, \mathrm{ad}^0(\bar{\rho})^*)$.

Proposition 3.7 gives a way to enlarge S and \mathcal{D}_S , allowing ramification subject to Ramakrishna's deformation condition at the new places that forces $H_{\mathcal{D}_S^\perp}^1(\Gamma_S, \mathrm{ad}^0(\bar{\rho})^*)$ to be zero. We review

Ramakrishna's deformation condition in §3.1. The places of K at which we define this condition are found using the Chebotarev density theorem: each additional place where we allow ramification subject to Ramakrishna's deformation condition decreases the dimension of the dual Selmer group. For such places to exist, we need non-zero classes in certain cohomology groups, whose existence relies on the local deformation conditions satisfying the inequality

$$\sum_{v \in S} \dim L_v \geq \sum_{v \in S} \dim H^0(\Gamma_v, \text{ad}^0(\bar{\rho})),$$

where L_v is the tangent space of the local deformation condition at v . Furthermore, $\bar{\rho}$ needs to be a “big” representation in the sense of Definition 3.4 in order to define Ramakrishna's deformation condition. Being a big representation is a more precise set of technical conditions that are implied for large enough p by the condition $G'(k) \subset \bar{\rho}(\Gamma_K)$ appearing in Theorem 3.12.

For the tangent space inequality to hold, it is crucial that $\bar{\rho}$ be an odd representation. The minimally ramified deformation conditions we will use at places v where $\bar{\rho}$ is ramified satisfy $\dim H^0(\Gamma_v, \text{ad}^0(\bar{\rho})) = \dim L_v$. Using the Fontaine-Laffaille deformation condition at places above p , the tangent space inequality becomes

$$[K : \mathbf{Q}](\dim G - \dim B) \geq \sum_{v|\infty} h^0(\Gamma_v, \text{ad}^0(\bar{\rho}))$$

where B is a Borel subgroup of G ; this can only be satisfied if K is totally real and $\bar{\rho}$ is odd.

In conclusion, to use the local-to-global principle, the residual representation must be odd and the deformation conditions we use at places above p and places where $\bar{\rho}$ ramifies must be liftable.

1.3. Minimally Ramified Deformation Condition. Let $\ell \neq p$ be primes, L be a finite extension of \mathbf{Q}_ℓ , and k a finite field of characteristic p . For a residual representation $\bar{\rho} : \Gamma_L \rightarrow G(k)$, Ramakrishna's method requires a “nice” deformation condition for $\bar{\rho}$. If $\bar{\rho}$ were unramified, the unramified deformation condition would work. The interesting case is when $\bar{\rho}$ is ramified: we would like to define a deformation condition of lifts which are “ramified no worse than $\bar{\rho}$,” so the resulting deformation condition is liftable despite the fact that the unrestricted deformation condition for $\bar{\rho}$ may not be liftable. To be precise, we require a deformation condition that is liftable *and* whose tangent space has dimension (at least) $\dim_k H^0(\Gamma_L, \text{ad}^0(\bar{\rho}))$.

In the case that $G = \text{GL}_n$, the minimally ramified deformation condition defined in [CHT08, §2.4.4] works. We will generalize this to a *minimally ramified deformation condition* for symplectic and orthogonal groups when $p > n$. Attempting to generalize the argument of [CHT08, §2.4.4] to groups besides GL_n leads to a deformation condition based on parabolics which is *not* liftable. Instead, inspired by the arguments of [Tay08, §3] we define a deformation condition for symplectic and orthogonal groups based on deformations of a nilpotent element of $\mathfrak{g}_k = \text{Lie } G_k$.

Let us first review the minimally ramified deformation condition introduced for GL_n in [CHT08, §2.4.4]. The first step is to reduce to studying certain tamely ramified representations. Recall that Γ_L^t , the Galois group of the maximal tamely ramified extension of L , is isomorphic to the semi-direct product

$$\hat{\mathbf{Z}} \ltimes \prod_{p' \neq \ell} \mathbf{Z}_{p'}$$

where $\hat{\mathbf{Z}}$ is generated by a Frobenius ϕ for L and the conjugation action by ϕ on each $\mathbf{Z}_{p'}$ is given by the p' -adic cyclotomic character. We consider tamely ramified representations which factor through the quotient $\hat{\mathbf{Z}} \ltimes \mathbf{Z}_p$ (recall $p \neq \ell$). Picking a topological generator τ for \mathbf{Z}_p , the action is explicitly given by

$$\phi \tau \phi^{-1} = q \tau$$

where q is the size of the residue field of L . Note q is a power of ℓ , so it is relatively prime to p . Arguments in [CHT08] reduce the lifting problem to studying representations of the group

$T_q := \widehat{\mathbf{Z}} \ltimes \mathbf{Z}_p$. This reduction generalizes without surprises to symplectic and orthogonal groups in §7 (but the argument is genuinely restricted to orthogonal and symplectic groups as it relies heavily on the pairing).

The second step is to specify when a lift of $\bar{\rho} : T_q \rightarrow \mathrm{GL}_n(k)$ is “ramified no worse than $\bar{\rho}$ ”. For a coefficient ring R , a deformation $\rho : T_q \rightarrow \mathrm{GL}_n(R)$ is *minimally ramified* according to [CHT08] when the natural k -linear map

$$(1.2) \quad \ker((\rho(\tau) - 1_n)^i) \otimes_R k \rightarrow \ker((\bar{\rho}(\tau) - 1_n)^i)$$

is an isomorphism for all i . The deformation condition is analyzed as follows:

- defining $V_i = \ker((\bar{\rho}(\tau) - 1_n)^i)$ gives a flag

$$0 \subset V_r \subset V_{r-1} \subset \dots \subset V_1 \subset k^n.$$

This flag determines a parabolic k -subgroup $\bar{P} \subset \mathrm{GL}_n$ (points which preserve the flag) such that $\bar{\rho}(\tau) \in (\mathcal{R}_u \bar{P})(k)$ and $\bar{\rho}(\phi) \in \bar{P}(k)$;

- lift \bar{P} to a parabolic subgroup P of GL_n . The deformation functor of such lifts is formally smooth, and for any minimally ramified deformation ρ over R there is a choice of such P for which $\rho(\tau) \in (\mathcal{R}_u P)(R)$ and $\rho(\phi) \in P(R)$. Conversely, any ρ with this property is minimally ramified;
- Finally, for the standard block-upper-triangular choice of P , one shows the deformation functor

$$\{(T, \Phi) : T \in \mathcal{R}_u P, \Phi \in P, \Phi T \Phi^{-1} = T^q, \bar{T} = \bar{\rho}(\tau), \bar{\Phi} = \bar{\rho}(\phi)\}$$

is formally smooth by building the universal lift over a power series ring; this uses explicit calculations with block-upper-triangular matrices.

To generalize beyond GL_n , we need to replace (1.2) with a more group-theoretic criterion. The naive generalization is to associate a parabolic \bar{P} to $\bar{\rho}$ and then use the following definition.

Definition 1.4. For a coefficient ring R , say a lift $\rho : T_q \rightarrow G(R)$ is *ramified with respect to \bar{P}* provided that there exists a parabolic R -subgroup $P \subset G_R$ lifting \bar{P} such that $\rho(\tau) \in (\mathcal{R}_u P)(R)$ and $\rho(\phi) \in P(R)$.

This idea does not work. Let us focus on the symplectic case to illustrate what goes wrong.

The first problem is to associate a parabolic subgroup to $\bar{\rho}$. Recall that parabolic subgroups of a symplectic group correspond to isotropic flags $0 \subset V_1 \subset \dots \subset V_r \subset V_r^\perp \subset \dots \subset V_1^\perp \subset k^{2n}$. There is no reason that the flag determined by (1.2) is isotropic, so we would need some other method of producing a parabolic \bar{P} such that $\bar{\rho}(\tau) \in (\mathcal{R}_u \bar{P})(k)$. In [BT71], Borel and Tits give a natural way to associate to the unipotent $\bar{\rho}(\tau)$ a smooth connected unipotent k -subgroup of G . The normalizer of this subgroup is always parabolic and so gives a candidate for \bar{P} . However, working out examples in GL_n for small n shows that this produces a different parabolic than the one determined by (1.2). This raises the natural question of how sensitive the smoothness of the deformation condition is to the choice of parabolic.

This leads to the second, larger problem: there are examples such that for *every* parabolic \bar{P} satisfying $\bar{\rho}(\tau) \in (\mathcal{R}_u \bar{P})(k)$, not all deformations ramified with respect to \bar{P} are liftable.

Example 1.5. Take $L = \mathbf{Q}_{29}$ and $k = \mathbf{F}_7$. Consider the representation $\bar{\rho} : T_{29} \simeq \widehat{\mathbf{Z}} \ltimes \mathbf{Z}_7 \rightarrow \mathrm{GSp}_4(\mathbf{F}_7)$ defined by

$$\bar{\rho}(\tau) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \bar{\rho}(\phi) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

The deformation condition of lifts ramified relative to a parabolic \overline{P} of GSp_4 whose unipotent radical contains $\overline{\rho}(\tau)$ is not liftable for any choice of \overline{P} : there are lifts to the dual numbers that do not lift to $\mathbf{F}_7[\epsilon]/(\epsilon^3)$. This is easy to check with a computer algebra system such as [SAGE], since the existence of lifts can be reduced to a problem in linear algebra.

This latter problem is a general phenomenon, which we will explain conceptually in terms of Richardson orbits in §6.4.

The correct approach is to define a lift $\rho : T_q \rightarrow G(R)$ to be minimally ramified if $\rho(\tau)$ has “the same unipotent structure” as $\overline{\rho}(\tau)$. It is more convenient to work with nilpotent elements, using the exponential and logarithm maps (defined for nilpotent and unipotent elements since $p > n$). We wish to study lifts of the nilpotent $\overline{N} = \log(\overline{\rho}(\tau))$ to $N \in \mathfrak{g}$ that “remain nilpotent of the same nilpotent type as \overline{N} ”.

In §6.1, we make this notion of “same nilpotent type” rigorous. There are combinatorial parametrization of nilpotent orbits of algebraic groups over an algebraically closed field, for example in terms of partitions or root data, which make precise the notion that the values of $N \in \mathfrak{g}_{\mathcal{O}}$ in the special and generic fiber lie in nilpotent orbits with the same combinatorial data. For each nilpotent orbit σ , we use the results of §4 to choose particular elements $N_{\sigma} \in \mathfrak{g}_{\mathcal{O}}$ with this property lifting $\overline{N} \in \mathfrak{g}_k$. For a coefficient ring R , we define the “pure nilpotents” lifting \overline{N} to be the $\widehat{G}(R)$ -conjugates of N_{σ} .

Example 1.6. For example, let $G = \mathrm{GL}_3$ and

$$\overline{N} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Consider the lifts

$$N_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g} \quad \text{and} \quad N_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}.$$

Both are nilpotent under the embedding of \mathcal{O} into its fraction field K . The images of N_1 in \mathfrak{g}_K and \mathfrak{g}_k both lie in the nilpotent orbit corresponding to the partition $2+1$, so N_1 is an example of the type of nilpotent lift we want to consider. On the other hand, the image of N_2 in \mathfrak{g}_K lies in the nilpotent orbit corresponding to the partition 3 , while the image on \mathfrak{g}_k lies in the orbit corresponding to $2+1$, so we do not want to use it. The pure nilpotents lifting \overline{N} are $\widehat{G}(R)$ -conjugates of N_1 .

We then define a lift $\rho : T_q \rightarrow G(R)$ to be *minimally ramified* provided $\rho(\tau)$ is the exponential of a pure nilpotent lifting $\log \overline{\rho}(\tau) = \overline{N}$. Proposition 6.13 shows that this deformation condition is liftable. The main technical fact needed to analyze this deformation condition is that the scheme-theoretic centralizer $Z_G(N_{\sigma})$ is smooth over \mathcal{O} for N_{σ} as above. The smoothness of such centralizers over algebraically closed fields is well-understood, and in §5 we study $Z_G(N_{\sigma})$ and show that $Z_G(N_{\sigma})$ is *flat* over \mathcal{O} and hence smooth. Lemma 5.4 gives a criterion for flatness that is easy to verify for classical groups which suffices for our applications. We can reduce checking \mathcal{O} -flatness to the problem of finding elements $g \in Z_G(N_{\sigma})(\mathcal{O})$ such that g_k lies in any specified component of $Z_{G_k}(\overline{N})/Z_{G_k}(\overline{N})^{\circ}$. There are difficulties beyond the classical case due to the varied structure of $\pi_0(Z_G(\overline{N})_k)$ in general.

Remark 1.7. It is a fortuitous coincidence (for [CHT08]) that for GL_n the lifts minimally ramified in the preceding sense are exactly the lifts ramified with respect to a parabolic subgroup of G . This rests on the fact that all nilpotent orbits of GL_n are Richardson orbits (see §6.4 for details).

1.4. Fontaine-Laffaille Deformation Condition. Let K be a finite unramified extension of \mathbf{Q}_p , and \mathcal{O} be the ring of integers of a p -adic field L with residue field k such that L splits K over \mathbf{Q}_p . (The latter is always possible after extending k .) To produce geometric deformations, Ramakrishna's method requires a deformation condition $\mathcal{D}_{\bar{\rho}}$ for the residual representation $\bar{\rho} : \Gamma_K \rightarrow G(k)$ such that:

- $\mathcal{D}_{\bar{\rho}}$ is liftable;
- $\mathcal{D}_{\bar{\rho}}$ is large enough, in the precise sense that its tangent space has dimension

$$[K : \mathbf{Q}_p](\dim G - \dim B) + \dim_k H^0(\Gamma_K, \mathrm{ad}^0(\bar{\rho}))$$

where B is a Borel subgroup of G ;

- $\mathcal{D}_{\bar{\rho}}(\mathcal{O})$ consists of certain lattices in crystalline representations.

We construct such a condition using Fontaine-Laffaille theory.

Fontaine-Laffaille theory, introduced in [FL82], provides a way to describe torsion-crystalline representations with Hodge-Tate weights in an interval of length $p - 2$ in terms of semi-linear algebra when p is unramified in K . In particular, it provides an exact, fully faithful functor T_{cris} from the category of filtered Dieudonné modules to the category of $\mathcal{O}[\Gamma_K]$ -modules with continuous action, and describes the image (Fact 8.10). In [CHT08, §2.4.1], it is used to define a deformation condition for GL_n , where the allowable deformations of $\bar{\rho}$ are exactly the deformations of the corresponding Fontaine-Laffaille module. This requires the technical assumption that the representation $\bar{\rho}$ is torsion-crystalline with Hodge-Tate weights in an interval of length $p - 2$, and the important assumption that the Fontaine-Laffaille weights of $\bar{\rho}$ under each embedding of K into L are distinct (see Remark 8.17).

We will adapt these ideas to symplectic and orthogonal groups under the assumption that the Fontaine-Laffaille weights lie in an interval of length $\frac{p-2}{2}$. For symplectic groups and $K = \mathbf{Q}_p$, this was addressed in Patrikis's undergraduate thesis [Pat06]: we generalize this, and record proofs as the thesis is not readily available. The key idea is to introduce a symmetric or alternating pairing into the semi-linear algebra data. To do so, it is necessary to use (at least implicitly via statements about duality) the fact that the functor T_{cris} is compatible with tensor products. This requires the stronger assumption that the Fontaine-Laffaille weights lie in an interval of length $\frac{p-2}{2}$, which guarantees that the Fontaine-Laffaille weights of the tensor product lie in an interval of length $p - 2$. Furthermore, it is crucial to use the covariant version of the Fontaine-Laffaille functor used in [BK90] instead of the contravariant version studied in [FL82] in order for the compatibility with tensor products to hold. For more details, see §8.2. Given this, it is then reasonably straightforward to check that T_{cris} is compatible with duality and hence to translate the (perfect) alternating or symmetric pairing of Galois representations into a (perfect) symmetric or alternating pairing of Fontaine-Laffaille modules.

For a coefficient ring R , define $D_{\bar{\rho}}^{\mathrm{FL}}(R)$ to be all representations $\rho : \Gamma_K \rightarrow G(R)$ lifting $\bar{\rho}$ and lying in the essential image of T_{cris} . To study this Fontaine-Laffaille deformation condition, it suffices to study Fontaine-Laffaille modules. In particular, to show that the deformation condition is liftable (i.e. that it is always possible to lift a deformation satisfying the condition through a square-zero extension), it suffices to show that a Fontaine-Laffaille module with distinct Fontaine-Laffaille weights together with a perfect symmetric or skew-symmetric pairing can always be lifted through a square zero extension. This is a complicated but tractable problem in semi-linear algebra: Proposition 9.8 shows this is always possible. It is relatively simple to lift the underlying filtered module and the pairing, and requires more care to lift the semi-linear maps $\varphi_M^i : M^i \rightarrow M$. Likewise, to understand the tangent space of the deformation condition it suffices to study deformations of the Fontaine-Laffaille module corresponding to $\bar{\rho}$ to the dual numbers. Again, the most involved step is understanding possible lifts of the semi-linear maps after choosing a lift of the filtration and the pairing.

Remark 1.8. The proof that $D_{\bar{\rho}}^{\text{FL}}$ is liftable and the computation of the dimension of its tangent space both use in an essential way the hypothesis that for each embedding of K into L the Fontaine-Laffaille weights are pairwise distinct.

Remark 1.9. An alternative deformation condition to use at primes above p is a deformation condition based on the concept of an ordinary representation. This is studied for any connected reductive group in [Pat15, §4.1]. It is suitable for use in Ramakrishna's method, and can give lifting results for a different class of torsion-crystalline representations.

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2. DEFORMATIONS OF GALOIS REPRESENTATIONS

2.1. Algebraic Groups and Very Good Primes. Let \mathcal{O} be a discrete valuation ring with residue field k of characteristic p . Let G be smooth separated group scheme over \mathcal{O} such that the identity components of the fibers are reductive.¹ Then G° is a reductive \mathcal{O} -subgroup scheme of G and G/G° is a separated étale \mathcal{O} -group scheme of finite presentation [Con14, Proposition 3.1.3 and Theorem 5.3.5]. Furthermore, by a result of Raynaud G is affine as it is a flat, separated, and of finite type with affine generic fiber over the discrete valuation ring \mathcal{O} [PY06, Proposition 3.1]. Call such G almost-reductive group schemes over \mathcal{O} . We say G is split if G° is split.

Remark 2.1. A reductive group scheme has connected fibers by definition: see [Con14, Definition 3.1.1], going back to [SGA3, XIX, 2.7]). Connectedness is important as in general the component group may jump across fibers. We wish to be able to work with GO_m which may have two connected components, so we work in this generality.

Let Φ a reduced and irreducible root system, and $P = (\mathbf{Z}\Phi^\vee)^*$ the weight lattice for Φ . We recall the notion of a very good prime.

Definition 2.2. The prime p is *good* for Φ provided that $\mathbf{Z}\Phi/\mathbf{Z}\Phi'$ is p -torsion free for all subsets $\Phi' \subset \Phi$. A good prime is *very good* provided that $P/\mathbf{Z}\Phi'$ is p -torsion free for all subsets $\Phi' \subset \Phi$. A prime is *bad* if it is not good.

Likewise, we say a prime p is *good* (or *very good*) for a general reduced root system if it is good (or very good) for each irreducible component. A prime p is *good* (or *very good*) for G provided it is good (or very good) for the root system of G_k° . For example, if $G = \text{GSp}_{2n}$ or $G = \text{GO}_m$ every prime except 2 is very good. The prime p being very good for a split almost-reductive group scheme G for example implies that:

- the center of $\text{Lie } G_k$ is the Lie algebra of Z_{G_k} , and $\text{Lie } G_k$ is a direct sum of $\text{Lie } G'_k$ and $\text{Lie } Z_{G_k}$, where G' is the derived group of G° and Z_{G_k} is the center of G_k° ;
- $Z_{G'_k}$ and $\pi_1(G_k^\circ)$ have order prime to p .

These facts are well-known.

2.2. Deformation Functors. Next we recall some facts about the deformation theory for Galois representations: a basic reference is [Maz97], with the extension to algebraic groups beyond GL_n discussed in [Til96].

Let Γ be a pro-finite group satisfying the following finiteness property: for every open subgroup $\Gamma_0 \subset \Gamma$, there are only finitely many continuous homomorphisms from Γ_0 to $\mathbf{Z}/p\mathbf{Z}$. This is true for

¹For results about reductive group schemes, we refer to [Con14] which gives a self-contained development, using more recent methods, of results from [SGA3].

the absolute Galois group of a local field and for the Galois group of the maximal extension of a number field unramified outside a finite set of places.

Let $\widehat{\mathcal{C}}_{\mathcal{O}}$ be the category of coefficient \mathcal{O} -algebras: complete local Noetherian rings with residue field k , with morphisms local homomorphisms inducing the identity map on k and with the structure morphism a map of coefficient rings. Let $\mathcal{C}_{\mathcal{O}}$ denote the full subcategory of Artinian coefficient \mathcal{O} -algebras. Recall that a *small* surjection of coefficient \mathcal{O} -algebras $f : A_1 \rightarrow A_0$ is a surjection such that $\ker(f) \cdot \mathfrak{m}_{A_1} = 0$.

For $A \in \widehat{\mathcal{C}}_{\mathcal{O}}$, define

$$\widehat{G}(A) := \ker(G(A) \rightarrow G(k))$$

We are interested in deforming a fixed $\bar{\rho} : \Gamma \rightarrow G(k)$. Let $\mathfrak{g} = \text{Lie } G$.

- Let $f : A_1 \rightarrow A_0$ be a morphism in $\widehat{\mathcal{C}}_{\mathcal{O}}$ and $\rho_0 : \Gamma \rightarrow G(A_0)$ a continuous homomorphism. A *lift* of ρ_0 to A_1 is a continuous homomorphism $\rho_1 : \Gamma \rightarrow G(A_1)$ such that the following diagram commutes:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\rho_1} & G(A_1) \\ & \searrow \rho_0 & \downarrow f \\ & & G(A_0) \end{array}$$

Define the functor $D_{\bar{\rho}, \mathcal{O}}^{\square} : \widehat{\mathcal{C}}_{\mathcal{O}} \rightarrow \text{Sets}$ by sending a coefficient \mathcal{O} -algebra A to the set of lifts of $\bar{\rho}$ to A .

- With the notation above, two lifts ρ and ρ' of $\bar{\rho}$ to $A_1 \in \mathcal{C}_{\mathcal{O}}$ are *strictly equivalent* if they are conjugate by an element of $\widehat{G}(A_1)$. A *deformation* of ρ_0 to A_1 is a strict equivalence class of lifts. Define the functor $D_{\bar{\rho}, \mathcal{O}} : \widehat{\mathcal{C}}_{\mathcal{O}} \rightarrow \text{Sets}$ by sending a coefficient \mathcal{O} -algebra A to the set of deformations of $\bar{\rho}$ to A .

We will drop the subscript \mathcal{O} when it is clear from context.

Fact 2.3. *The functor $D_{\bar{\rho}, \mathcal{O}}^{\square}$ is representable. When $\mathfrak{g}_k^{\Gamma} = \text{Lie}(Z_G)_k$, the functor $D_{\bar{\rho}, \mathcal{O}}$ is representable.*

The first part is simple, the second is a reformulation of [Til96, Theorem 3.3].

The representing objects are denoted $R_{\bar{\rho}, \mathcal{O}}^{\square}$ and (when it exists) $R_{\bar{\rho}, \mathcal{O}}$. While we usually care about deformations, it is technically easier to work with lifts.

This deformation theory is controlled by Galois cohomology. Let $\text{ad}(\bar{\rho})$ denote the representation of Γ on \mathfrak{g}_k via the adjoint representation. Letting G' be the derived subgroup of G° with Lie algebra \mathfrak{g}' , we also consider the representation $\text{ad}^0(\bar{\rho})$ of Γ on \mathfrak{g}'_k . As p is very good, we have $\mathfrak{g}_k = \mathfrak{g}'_k \oplus \mathfrak{z}_{\mathfrak{g}}$ where $\mathfrak{z}_{\mathfrak{g}}$ is the Lie algebra of Z_G . The condition in Fact 2.3 is just that $H^0(\Gamma, \text{ad}(\bar{\rho})) = \mathfrak{z}_{\mathfrak{g}}$, or equivalently that $H^0(\Gamma, \text{ad}^0(\bar{\rho})) = 0$. In general, since p is very good the natural map $H^i(\Gamma, \text{ad}^0(\bar{\rho})) \rightarrow H^i(\Gamma, \text{ad}(\bar{\rho}))$ is injective for all i ; we often use this without comment.

We can use the first order exponential map [Til96, §3.5] to understand the tangent space. Recall that for a smooth \mathcal{O} -group scheme G , and a small surjection $f : A \rightarrow A/I$ of coefficient rings ($I \cdot \mathfrak{m}_A = 0$), smoothness gives an isomorphism

$$\exp : \mathfrak{g} \otimes_k I \simeq \ker(G(A) \rightarrow G(A/I)) = \ker(\widehat{G}(A) \rightarrow \widehat{G}(A/I)).$$

The tangent space $D_{\bar{\rho}, \mathcal{O}}(k[\epsilon]/\epsilon^2)$ is identified with $H^1(\Gamma, \text{ad}(\bar{\rho}))$: Under this isomorphism, the cohomology class of a 1-cocycle τ corresponds to the lift $\rho(g) = \exp(\epsilon\tau(g))\bar{\rho}(g)$. For the framed deformation ring $R_{\bar{\rho}, \mathcal{O}}^{\square}$, the tangent space is identified with the k -vector space $Z^1(\Gamma, \text{ad}(\bar{\rho}))$ of (continuous) 1-cocycles of Γ valued in $\text{ad}(\bar{\rho})$.

Remark 2.4. We also observe that

$$\dim_k Z^1(\Gamma, \text{ad}(\bar{\rho})) - \dim_k H^1(\Gamma, \text{ad}(\bar{\rho})) = \dim_k B^1(\Gamma, \text{ad}(\bar{\rho})) = \dim_k \mathfrak{g} - \dim_k H^0(\Gamma, \text{ad}(\bar{\rho}))$$

since the space of coboundaries admits a surjection from $\mathrm{ad}(\bar{\rho})$ with kernel $\mathrm{ad}(\bar{\rho})^\Gamma$. This will be useful when comparing dimensions of framed and unframed deformation rings that are smooth.

We will want to studying special classes of deformations.

Definition 2.5. A *lifting condition* is a sub-functor $\mathcal{D}^\square \subset D_{\bar{\rho}, \mathcal{O}}^\square : \mathcal{C}_{\mathcal{O}} \rightarrow \mathrm{Sets}$ such that:

- (1) For any coefficient ring A , $\mathcal{D}^\square(A)$ is closed under strict equivalence.
- (2) Given a Cartesian diagram in $\mathcal{C}_{\mathcal{O}}$

$$\begin{array}{ccc} A_1 \times_{A_0} A_2 & \xrightarrow{\pi_2} & A_2 \\ \downarrow \pi_1 & & \downarrow \\ A_1 & \longrightarrow & A_0 \end{array}$$

and $\rho \in D_{\bar{\rho}, \mathcal{O}}^\square(A_1 \times_{A_0} A_2)$, we have $\rho \in \mathcal{D}^\square(A_1 \times_{A_0} A_2)$ if and only if $\mathcal{D}^\square(\pi_1) \circ \rho \in \mathcal{D}^\square(A_1)$ and $\mathcal{D}^\square(\pi_2) \circ \rho \in \mathcal{D}^\square(A_2)$.

As it is closed under strict equivalence, we naturally obtain a *deformation condition*, a sub-functor $\mathcal{D} \subset D_{\bar{\rho}, \mathcal{O}}$.

By Schlessinger's criterion [Sch68, Theorem 2.11] being a lifting condition is equivalent to the functor \mathcal{D}^\square being pro-representable. Likewise, the deformation condition \mathcal{D} associated to a lifting condition \mathcal{D}^\square is pro-representable provided that $D_{\bar{\rho}, \mathcal{O}}$ is.

The tangent space of a deformation condition \mathcal{D} is a k -subspace of $H^1(\Gamma, \mathrm{ad}(\bar{\rho}))$, and will be denoted by $H_{\mathcal{D}}^1(\Gamma, \mathrm{ad}(\bar{\rho}))$. For a small surjection $A_1 \rightarrow A_0$ and $\rho \in \mathcal{D}(A_0)$, the set of deformations of ρ to A_1 subject to \mathcal{D} is a $H_{\mathcal{D}}^1(\Gamma, \mathrm{ad}(\bar{\rho}))$ -torsor. This torsor-structure is compatible with the action of the unrestricted tangent space to $D_{\bar{\rho}}$ on the space of all deformations of ρ to A_1 .

Example 2.6. Let G' be the derived group of G° . The most basic examples of deformation conditions are the conditions imposed by fixing the lift of the homomorphism $\Gamma \rightarrow (G/G')(k)$. To be precise, for the quotient map $\mu : G \rightarrow G/G' =: S$, a fixed $\nu : \Gamma \rightarrow S(\mathcal{O})$ lifting $\mu \circ \bar{\rho}$, and $A \in \widehat{\mathcal{C}}_{\mathcal{O}}$ with structure morphism $\iota : \mathcal{O} \rightarrow A$, we define a deformation condition $\mathcal{D}_0 \subset \mathcal{D}_{\bar{\rho}}$ by

$$\mathcal{D}_\nu(A) = \{\rho \in \mathcal{D}_{\bar{\rho}}(A) \mid \Gamma \rightarrow G(A) : \mu_A \circ \rho = \iota \circ \nu_A\}.$$

One checks this is a deformation condition. Its tangent space is $H^1(\Gamma, \mathrm{ad}^0(\bar{\rho}))$ since p is very good. We define $\mathcal{D}_{\bar{\rho}}^\square$ similarly.

Another important easy example is the *unramified* deformation condition for a non-archimedean place v where ρ is unramified: this consists of lifts that are unramified (possibly with a specified choice of ν). The tangent space is $H_{\mathrm{nr}}^1(\Gamma_v, \mathrm{ad}(\bar{\rho}))$ (respectively $H_{\mathrm{nr}}^1(\Gamma_v, \mathrm{ad}^0(\bar{\rho}))$).

Definition 2.7. A deformation condition \mathcal{D} is *locally liftable* (over \mathcal{O}) if for all small surjections $f : A_1 \rightarrow A_0$ of coefficient \mathcal{O} -algebras the natural map

$$\mathcal{D}(f) : \mathcal{D}(A_1) \rightarrow \mathcal{D}(A_0)$$

is surjective.

A geometric way to check local liftability is to show that the corresponding deformation ring (when it exists) is smooth. Obviously it suffices to check liftability for lifts instead of deformations, so we can work with the framed deformation ring and avoid representability issues for $\mathcal{D}_{\bar{\rho}}$.

Example 2.8. The unramified deformation condition is liftable: an unramified lift is completely determined by the image of Frobenius in $G(A_0)$, and G is smooth over \mathcal{O} .

When attempting to lift with a fixed lift ν of $\Gamma \rightarrow (G/G')(k)$, the obstruction to lifting is measured by a 2-cocycle $\mathrm{ob}(\rho_0)$ that lies in $H^2(\Gamma, \mathrm{ad}^0(\bar{\rho}))$. To see this, recall that the obstruction cocycle is defined by picking a set theoretic lift ρ_1 of a given $\rho_0 : \Gamma_K \rightarrow G(A_0)$: the 2-cocycle records

the failure of ρ_1 to be a homomorphism. By choosing the continuous set-theoretic lift $\Gamma_K \rightarrow G(A_1)$ so that $\Gamma_K \rightarrow (G/G')(A_0)$ agrees with ν (as we may easily do since $\ker \rho_0$ is open in Γ_K), the obstruction cocycle takes values in $\mathrm{ad}^0(\bar{\rho})$.

2.3. Global Deformations. We now study global deformation conditions. Let K be a number field, S a finite set of places of K that contains all the places of K at which $\bar{\rho}$ are ramified and all archimedean places. Let Γ_S be the Galois group of the maximal extension of K unramified outside of S and Γ_K be the absolute Galois group of K .

Definition 2.9. A *global deformation condition* \mathcal{D}_S for $\bar{\rho} : \Gamma_S \rightarrow G(k)$ is a collection of local deformation conditions $\{\mathcal{D}_v\}_{v \in S}$ for $\bar{\rho}|_{\Gamma_v}$. We say it is *locally liftable* (over \mathcal{O}) if each \mathcal{D}_v is locally liftable (over \mathcal{O}). A *global deformation of $\bar{\rho} : \Gamma_S \rightarrow G(k)$ subject to \mathcal{D}_S* is a deformation $\rho : \Gamma_S \rightarrow G(A)$ such that $\rho|_{\Gamma_v} \in \mathcal{D}_v(A)$ for all $v \in S$.

For $v \in S$, let L_v denote the tangent space of the local deformation condition \mathcal{D}_v . A global deformation condition gives a generalized Selmer group. We will be mainly interested in the *dual Selmer group*

$$(2.1) \quad H_{\mathcal{D}_S^\perp}^1(\Gamma_S, \mathrm{ad}(\bar{\rho})^*) = \{x \in H^1(\Gamma_S, \mathrm{ad}(\bar{\rho})^*) : \mathrm{res}_v(x) \in L_v^\perp \text{ for all } v \in S\}.$$

For Ramakrishna's method to work, it is crucial that the local tangent spaces be large enough relative to the local invariants. We say that a global deformation condition satisfies the *tangent space inequality* if

$$(2.2) \quad \sum_{v \in S} \dim L_v \geq \sum_{v \in S} \dim H^0(\Gamma_v, \mathrm{ad}^0(\bar{\rho})).$$

Let $\mathcal{D}_S = \{\mathcal{D}_v\}$ be a global deformation condition, and G' be the derived group of G° with quotient $\mu : G \rightarrow G/G'$. We assume that the deformation condition includes the condition of fixing a lift $\nu : \Gamma_K \rightarrow (G/G')(\mathcal{O})$ of the character $\mu \circ \bar{\rho} : \Gamma_K \rightarrow (G/G')(k)$. This means that all of the local deformation conditions have tangent spaces lying in $H^1(\Gamma_v, \mathrm{ad}^0(\bar{\rho}))$, and the obstruction cocycles automatically land in $H^2(\Gamma_v, \mathrm{ad}^0(\bar{\rho}))$ (see Example 2.6 and Example 2.8), with similar statements for global deformation conditions. In favorable circumstances, we can use the following local-to-global principle to produce lifts.

Proposition 2.10. *Let $A_1 \rightarrow A_0$ be a small extension of coefficient \mathcal{O} -algebras with kernel I , and consider a lift $\rho_0 : \Gamma_S \rightarrow G(A_0)$ of $\bar{\rho}$ subject to \mathcal{D}_S . Provided $H_{\mathcal{D}_S^\perp}^1(\Gamma_S, \mathrm{ad}^0(\bar{\rho})^*) = 0$, lifting ρ_0 to A_1 subject to \mathcal{D}_S is equivalent to lifting $\rho_0|_{\Gamma_v}$ to A_1 subject to \mathcal{D}_v for all $v \in S$.*

Proof. One direction is obvious. Conversely, suppose we have local lifts. The key input is the Poitou-Tate exact sequence:

$$H^1(\Gamma_S, \mathrm{ad}^0(\bar{\rho})) \rightarrow \bigoplus_{v \in S} H^1(\Gamma_v, \mathrm{ad}^0(\bar{\rho}))/L_v \rightarrow H_{\mathcal{D}_S^\perp}^1(\Gamma_S, \mathrm{ad}^0(\bar{\rho})^*)^\vee \rightarrow H^2(\Gamma_S, \mathrm{ad}^0(\bar{\rho})) \rightarrow \bigoplus_{v \in S} H^2(\Gamma_v, \mathrm{ad}^0(\bar{\rho})).$$

The vanishing of $H_{\mathcal{D}_S^\perp}^1(\Gamma_S, \mathrm{ad}^0(\bar{\rho})^*)$ implies that the first map is a surjection and the last an injection.

As $\rho_0|_{\Gamma_v}$ is liftable for all $v \in S$, the local obstructions to lifting vanish. The global obstruction to lifting ρ_0 to A_1 , $\mathrm{ob}(\rho_0) \in H^2(\Gamma_S, \mathrm{ad}^0(\bar{\rho})) \otimes I$, therefore maps to 0 in $\bigoplus_{v \in S} H^2(\Gamma_v, \mathrm{ad}^0(\bar{\rho})) \otimes I$. As this latter restriction map is injective, there is a lift ρ_1 of ρ_0 to A_1 on Γ_S . We wish to show it can be chosen subject to \mathcal{D}_S .

The set of all lifts of $\rho_0|_{\Gamma_v}$ is an $H^1(\Gamma_v, \mathrm{ad}^0(\bar{\rho})) \otimes I$ -torsor. The existence of local lifts means that there exist $\phi_v \in H^1(\Gamma_v, \mathrm{ad}^0(\bar{\rho})) \otimes I$ such that $\phi_v \cdot \rho_0|_{\Gamma_v} \in \mathcal{D}_v(A_1)$. By the surjectivity of the first map in the sequence, there exists $\phi \in H^1(\Gamma_S, \mathrm{ad}^0(\bar{\rho})) \otimes I$ such that $\phi|_v$ agrees with ϕ_v up to an element of $L_v \otimes I$ for all $v \in S$. As the set of lifts of $\rho_0|_{\Gamma_v}$ subject to \mathcal{D}_v is a $L_v \otimes I$ -torsor, this implies that $(\phi \cdot \rho_1)|_{\Gamma_v} \in \mathcal{D}_v(A_1)$. In other words, $\phi \cdot \rho_1$ is a lift of ρ_0 to A_1 satisfying \mathcal{D}_S . \square

Remark 2.11. This is a variant of [Pat15, Corollary 3.11].

3. GENERALIZING RAMAKRISHNA'S METHOD

The key to generalizing Ramakrishna's method is the ability to choose local conditions so that Proposition 2.10 will apply. This generalization is carried for split reductive group schemes with connected fibers in [Pat15] and in the author's thesis with only minor technical differences between them. Here we refer to [Pat15] for proofs and only point out the modifications necessary to deal with split almost-reductive groups like GO_m . So let \mathcal{O} be the ring of integers in a p -adic field with residue field k , and let $q = \#k$. Consider a split almost-reductive group scheme G over \mathcal{O} with Lie algebra \mathfrak{g} . Let K be a number field and denote the p -adic cyclotomic character by $\chi : \Gamma_K \rightarrow \mathbf{Z}_p^\times$, with reduction $\bar{\chi} : \Gamma_K \rightarrow \mathbf{F}_p^\times$. Fix a split maximal torus $T \subset G^\circ$.

3.1. Ramakrishna's Deformation Condition. We start by assuming:

- (A1) there is $\gamma \in \Gamma_K$ such that $\bar{\rho}(\gamma) \in G^\circ(k)$ is regular semisimple, and $Z_{G_k}(\bar{\rho}(\gamma))^\circ = T_k$;
- (A2) there is a unique root $\alpha \in \Phi(G, T)$ such that $\alpha(\bar{\rho}(\gamma)) = \bar{\chi}(\gamma)$;
- (A3) there is a place v of K lying over a rational prime ℓ such that $\bar{\rho}$ is unramified at v , $\bar{\rho}(\Gamma_v) \subset G^\circ(k)$, and $\bar{\rho}(\mathrm{Frob}_v)$ is regular semisimple element. The identity component of $Z_{G_k}(\bar{\rho}(\mathrm{Frob}_v))$ is T_k , and $\alpha(\bar{\rho}(\mathrm{Frob}_v)) = \bar{\chi}(\mathrm{Frob}_v) = \bar{\chi}(\gamma) \neq 1$.

Under these assumptions, we can define Ramakrishna's deformation condition $\mathcal{D}_v^{\mathrm{ram}}$ for $\rho_v : \Gamma_v \rightarrow G^\circ(k)$ as in [Pat15, §4.2]. We form the root group $U_\alpha \subset G^\circ$ associated to α .

Definition 3.1. For a coefficient \mathcal{O} -algebra A , consider a lift $\rho : \Gamma_v^\dagger \rightarrow G^\circ(A)$. The lift ρ satisfies *Ramakrishna's condition relative to T* provided that $\rho(\mathrm{Frob}_v) \in T(A)$, $\alpha(\rho(\mathrm{Frob}_v)) = \chi(\mathrm{Frob}_v)$, and $\rho(\mathrm{Gal}(K_v^\dagger/K_v^{\mathrm{nr}})) \subset U_\alpha(A) \subset G^\circ(A)$.

Define *Ramakrishna's deformation condition* $\mathcal{D}_v^{\mathrm{ram}}(A)$ to be lifts which are $\widehat{G}(A)$ -conjugate to one which satisfies Ramakrishna's condition relative to T .

Letting S be the quotient of G° by its derived group with quotient map μ , we can also study lifts $\rho : \Gamma_{K_v} \rightarrow G^\circ(A)$ such that $\mu \circ \rho$ is a fixed unramified lift ν of $\mu \circ \bar{\rho}$. As the condition $\mu \circ \rho = \nu$ cuts out a closed subscheme of the universal lifting ring for $\mathcal{D}_v^{\mathrm{ram}}$, this is a deformation condition we will denote by $\mathcal{D}_v^{\mathrm{ram}, \nu}$.

Fact 3.2. *The deformation conditions $\mathcal{D}_v^{\mathrm{ram}}$ and $\mathcal{D}_v^{\mathrm{ram}, \nu}$ are liftable. The dimension of their tangent spaces are $\dim H^0(\Gamma_v, \mathrm{ad}(\bar{\rho}))$ and $\dim H^0(\Gamma_v, \mathrm{ad}^0(\bar{\rho}))$ respectively.*

Remark 3.3. In order to apply the results of [Pat15, §4.2] (or the analogous results in [Boo16, §2.4]), it is important to work with a connected reductive group. This is why we assume that $\bar{\rho}(\Gamma_v) \subset G^\circ(k)$. Similarly, when analyzing disconnected L -groups [Pat15, §9.2] reduces to situations where the Galois-representation on the (constant) component group scheme is trivial in order to use this deformation condition.

3.2. Big Representations. Let $K(\mathrm{ad}^0(\bar{\rho}))$ and $K(\mathrm{ad}^0(\bar{\rho})^*)$ denote the fixed field of the kernel of the actions of Γ_K on $\mathrm{ad}^0(\bar{\rho})$ and $\mathrm{ad}^0(\bar{\rho})^*$ respectively, and F be the compositum. Let \mathcal{D}_S be a global deformation condition satisfying the tangent space inequality (2.2). The natural class of representations $\bar{\rho} : \Gamma_K \rightarrow G(k)$ to which Ramakrishna's method will apply are those which satisfy the following conditions:

Definition 3.4. A representation $\bar{\rho} : \Gamma_K \rightarrow G(k)$ is *big* relative to \mathcal{D}_S provided that

- (i) we have $H^0(\Gamma_K, \mathrm{ad}^0(\bar{\rho})) = H^0(\Gamma_K, \mathrm{ad}^0(\bar{\rho})^*) = 0$;
- (ii) we have $H^1(\mathrm{Gal}(K(\mathrm{ad}^0(\bar{\rho}))/K), \mathrm{ad}^0(\bar{\rho})) = 0$ and $H^1(\mathrm{Gal}(K(\mathrm{ad}^0(\bar{\rho})^*)/K), \mathrm{ad}^0(\bar{\rho})^*) = 0$;
- (iii) for any non-zero $\psi \in H_{\mathcal{D}_S}^1(\Gamma_S, \mathrm{ad}^0(\bar{\rho}))$ and $\phi \in H_{\mathcal{D}_S}^1(\Gamma_S, \mathrm{ad}^0(\bar{\rho})^*)$, the fields F_ψ and F_ϕ are linearly disjoint over F , where F_ψ (respectively F_ϕ) is the fixed field of the kernel of the homomorphism obtained by restricting ψ (respectively ϕ) to Γ_F ;

- (iv) for any non-zero $\psi \in H_{\mathcal{D}_S}^1(\Gamma_S, \text{ad}^0(\bar{\rho}))$ and $\phi \in H_{\mathcal{D}_S}^1(\Gamma_S, \text{ad}^0(\bar{\rho})^*)$, there is an element $\gamma \in \Gamma_K$ such that $\bar{\rho}(\gamma) \in G^\circ(k)$ is regular semisimple with $Z_{G_k}(\bar{\rho}(\gamma))^\circ = T_k$, and for which there is a unique root $\alpha \in \Phi(G, T)$ satisfying $\alpha(\bar{\rho}(\gamma)) = \bar{\chi}(\gamma) \neq 1$, for which $k[\psi(\Gamma_K)]$ has an element with non-zero \mathfrak{t}_α -component, and for which $k[\phi(\Gamma_K)]$ has an element with non-zero $\mathfrak{g}_{-\alpha}$ -component.

Remark 3.5. In (iv), note that $\alpha(\bar{\rho}(\gamma))$ makes sense because $\bar{\rho}(\gamma) \in T(k)$, as any semisimple element $g \in G^\circ(k)$ satisfies $g \in Z_{G_k}(g)^\circ$. Also, \mathfrak{t}_α is the span of the α -coroot vector and $\mathfrak{g}_{-\alpha}$ is the $-\alpha$ root space.

Remark 3.6. Observe that these conditions are insensitive to extension of k .

Let S be a finite set of places of K containing the archimedean places, the places over p , and the places where $\bar{\rho}$ is ramified.

Proposition 3.7. *Let \mathcal{D}_S be a global deformation condition that satisfies the tangent space inequality, and suppose $\bar{\rho}$ is big relative to \mathcal{D}_S . There is a finite set of places $T \supset S$ such that the deformation condition \mathcal{D}_T obtained by extending \mathcal{D}_S allowing deformations according to $\mathcal{D}_v^{\text{ram}}$ for $v \in T \setminus S$ satisfies*

$$H_{\mathcal{D}_T}^1(\Gamma_K, \text{ad}^0(\bar{\rho})^*) = 0.$$

Proof. This is almost [Pat15, Proposition 5.2]. There, we find places v of K satisfying the hypotheses necessary to define Ramakrishna's deformation condition using the Chebotarev density theorem on the extension $F_\psi F_\phi K(\bar{\rho})/K$, where $K(\bar{\rho})$ is the fixed field of the kernel of $\bar{\rho}$, using as input the element γ in the definition of bigness. The only difference here is that we have the additional requirement that $\bar{\rho}(\Gamma_v) \subset G^\circ(k)$, or equivalently $\bar{\rho}(\text{Frob}_v) \in G^\circ(k)$ as $\bar{\rho}$ is unramified at v .

Let K' denote the fixed field of the kernel of the composition of $\bar{\rho}$ with the map to the component group of G_k . We apply the Chebotarev density theorem to the extension $F_\psi F_\phi K(\bar{\rho})/K'$, using that $\bar{\rho}(\gamma) \in G^\circ(k)$, obtaining a place v' with $\bar{\rho}(\text{Frob}_{v'}) \in G^\circ(k)$ as well as the original conditions. As the primes of K' which are split over K have density 1, we may freely add the condition that the place v' of K' is split over the place v of K . As $K'_{v'} = K_v$, we conclude that $\bar{\rho}(\text{Frob}_v) \in G^\circ(k)$. The original argument then shows that adding Ramakrishna's deformation condition at v to the global deformation condition decreases the size of the dual Selmer group. \square

There is an easy case in which we can check that $\bar{\rho}$ is big relative to a global deformation \mathcal{D}_S satisfying the tangent space inequality. Let G' be the derived group of G° , and h the Coxeter number of G' .

Proposition 3.8. *Suppose that $K \cap \mathbf{Q}(\mu_p) = \mathbf{Q}$, that p is relatively prime to the order of the component group of G_k , and that the root system of G° is irreducible and of rank greater than 1. If $G'(k) \subset \bar{\rho}(\Gamma_S)$, and $p-1$ is greater than the maximum of $8\#Z_{G'}$ and*

$$\begin{cases} (h-1)\#Z_{G'} & \text{if } \#Z_{G'} \text{ is even} \\ (2h-2)\#Z_{G'} & \text{if } \#Z_{G'} \text{ is odd} \end{cases}$$

then $\bar{\rho}$ is big relative to \mathcal{D}_S .

Proof. This is part of the proof of [Pat15, Theorem 6.4]. Small modifications are needed to deal with almost-reductive G . In particular, when deducing (ii), it is necessary to use inflation-restriction to pass from the statement that $H^1(G'(k), \text{ad}^0(\bar{\rho})) = 0$ to the statement that $H^1(\bar{\rho}(\Gamma_S), \text{ad}^0(\bar{\rho})) = 0$ using that the index of $G'(k) \subset G^\circ(k) \subset G(k)$ is prime to p . The arguments for (iii) and (iv) are unchanged: both rely on constructing elements in the image of $\bar{\rho}$ using root data, so the argument can take place inside G° . \square

Remark 3.9. The argument is not optimized to produce the weakest restriction on p . The approach works uniformly for any irreducible root system: in any specific case improvements should be possible.

Remark 3.10. The formulation in [Boo16, §2.3] is very similar (only treating the case that G has connected fibers). Conditions (i), (iii), and (iv) are replaced by the simpler but stronger conditions that $\mathrm{ad}^0(\bar{\rho})$ is an absolutely irreducible representation of Γ_K and the condition that

- (iii) there exists $\gamma \in \Gamma_K$ such that $\bar{\rho}(\gamma)$ is regular semisimple with associated maximal torus $Z_{G_k}(\bar{\rho}(\gamma))^\circ$ equal to the split maximal torus T_k , and for which there is a unique root $\alpha \in \Phi(G, T)$ satisfying $\alpha(\bar{\rho}(\gamma)) = \bar{\chi}(\gamma) \neq 1$. (If $\dim T = 1$, we furthermore require that $\bar{\chi}(\gamma)^3 \neq 1$.)

This condition holds in the situation of Proposition 3.8. The analysis follows analogous lines. The conditions that the root system of G° is irreducible and that G' is not of rank 1 are removed by additional bookkeeping and imposing a stronger bound on p when the rank of G' is 1.

3.3. Choosing Deformation Conditions. Let G' be the derived group of G° and $\mu : G \rightarrow G/G'$ be the quotient map. For a fixed lift ν of

$$\mu \circ \bar{\rho} : \Gamma_K \rightarrow (G/G')(k),$$

the heart of the matter is to choose deformation conditions so that we may apply Proposition 2.10 and Proposition 3.7 to produce a geometric lift of $\bar{\rho}$ with $\mu \circ \bar{\rho} = \nu$. We need:

- (1) Locally liftable deformation conditions at finite places away from p where $\bar{\rho}$ is ramified.
- (2) Locally liftable deformation conditions at places above p whose characteristic-zero points are lattices in crystalline (or semistable representations).
- (3) The tangent space inequality (2.2) to hold, which will require $\bar{\rho}$ to be odd.

It is necessary to extend \mathcal{O} and k in order to define some of these deformation conditions: the condition that $\bar{\rho}$ is big is unaffected (Remark 3.6), so we are free to do so. We will find such deformation conditions when $G = \mathrm{GSp}_m$ with even $m \geq 4$ or $G = \mathrm{GO}_m$ with $m \geq 5$. In order to have the necessary oddness assumption on $\bar{\rho}$, in the latter case $m \not\equiv 2 \pmod{4}$.

At the places where $\bar{\rho}$ is ramified, in §6 and §7, we will construct a *minimally ramified deformation condition* by studying deformations of nilpotent (or equivalently unipotent) elements provided $p \geq m$. For each place, this will potentially require a finite extension of k . After such a further extension, this will be a liftable deformation condition at v with tangent space of dimension $h^0(\Gamma_v, \mathrm{ad}^0(\bar{\rho}))$ (see Corollary 7.16). This generalizes the results for GL_n obtained in [CHT08, §2.4.4].

At the places above p , when $G = \mathrm{GO}_m$ or GSp_m after extending k we will construct a *Fontaine-Laffaille deformation condition* using Fontaine-Laffaille theory in §9. This requires the assumption that $\nu \otimes \mathcal{O}[\frac{1}{p}]$ is crystalline, p is unramified in K , $\bar{\rho}$ is torsion-crystalline with Hodge-Tate weights in an interval of length $\frac{p-2}{2}$, and that the Fontaine-Laffaille weights for each \mathbf{Z}_p -embedding of \mathcal{O}_K into \mathcal{O} are pairwise distinct. The deformation condition is liftable, and the dimension of the tangent space will be $h^0(\Gamma_v, \mathrm{ad}^0(\bar{\rho})) + [K_v : \mathbf{Q}_p](\dim G_k - \dim B_k)$, where B is a Borel subgroup of G . This generalizes the results for GL_n obtained in [CHT08, §2.4.2].

Remark 3.11. The restriction that p is unramified in K and that the Hodge-Tate weights of $\bar{\rho}$ are in an interval of length $\frac{p-2}{2}$ is required to use Fontaine-Laffaille theory. Approaches using different flavors of integral p -adic Hodge theory should be able to remove it (for example, the deformation condition based on ordinary representations worked out by Patrikis [Pat15, §4.1] does so for a special class of representations). However, most previous work on studying deformation rings using integral p -adic Hodge theory only gives results about the crystalline deformation ring with p inverted, which does not suffice for our method.

The assumption that the Hodge-Tate weights are pairwise distinct is crucial, as otherwise the expected dimensions of the local crystalline deformation rings are too small to use in Ramakrishna's method.

We also need to specify a deformation condition at the archimedean places v : we just require lifts for which $\mu \circ \rho|_{\Gamma_v} = \nu|_{\Gamma_v}$. This condition is very simple to arrange, as $\#\Gamma_v \leq 2$. At a complex place, the dimension of the tangent space is zero and the dimension of the invariants is $\dim_k \mathrm{ad}^0(\bar{\rho})$. At a real place, the tangent space is zero when $p > 2$ and the invariants are the invariants of complex conjugation on $\mathrm{ad}^0(\bar{\rho})$.

Now we study the tangent space inequality (2.2). Let S be a set of places consisting of primes above p , places where $\bar{\rho}$ is ramified, and the archimedean places. When using the local deformation conditions as above at $v \in S$, the inequality (2.2) says exactly that

$$(3.1) \quad [K : \mathbf{Q}](\dim G_k - \dim B_k) = \sum_{v|p} [K_v : \mathbf{Q}_p](\dim G_k - \dim B_k) \geq \sum_{v|\infty} h^0(\Gamma_v, \mathrm{ad}^0(\bar{\rho})) = \sum_{v|\infty} \mathrm{ad}^0(\bar{\rho})^{\Gamma_v}$$

This is very strong: it is always true that $\dim \mathrm{ad}^0(\bar{\rho})^{\Gamma_v} \geq [K_v : \mathbf{R}](\dim G_k - \dim B_k)$, so (3.1) holds if and only if K is totally real and $\bar{\rho}$ is odd at all real places of K .

Assuming K is totally real and $\bar{\rho}$ is odd at all real places, we use Ramakrishna's deformation condition $\mathcal{D}_v^{\mathrm{ram}}$ at a collection of new places as in Proposition 3.7 (again possibly extending k). This gives a new deformation condition \mathcal{D}_T for which $H_{\mathcal{D}_T}^1(\Gamma_T, \mathrm{ad}^0(\bar{\rho})^*) = 0$. Using Proposition 2.10, we obtain the desired geometric lift.

Let us collect together all of our assumptions and record the result. For $G = \mathrm{GSp}_m$ with even $m \geq 4$ or $G = \mathrm{GO}_m$ with $m \geq 5$ and a big representation $\bar{\rho} : \Gamma_K \rightarrow G(k)$, fix a lift $\nu : \Gamma_K \rightarrow (G/G')(k)$ to \mathcal{O} of $\mu \circ \bar{\rho}$ such that $\nu \otimes \mathcal{O}[\frac{1}{p}]$ is Fontaine-Laffaille. We furthermore assume that K is totally real and that $\bar{\rho}$ is odd at all real places (which requires $m \not\equiv 2 \pmod{4}$). To use the Fontaine-Laffaille condition, we assume that p is unramified in K and that $\bar{\rho}$ is Fontaine-Laffaille at all places above p with Fontaine-Laffaille weights in an interval of length $\frac{p-2}{2}$, pairwise distinct for each \mathbf{Q}_p embedding of K into $\mathcal{O}[\frac{1}{p}]$. In order to use the minimally ramified deformation condition of §7, we require that $p \geq m$. We extend \mathcal{O} (and k) so that all of the required deformation conditions may be defined.

Theorem 3.12. *Under these conditions, there is a finite set T of places containing the archimedean places, the places above p , and the places where $\bar{\rho}$ is ramified such that there exists a lift $\rho : \Gamma_K \rightarrow G(\mathcal{O})$ such that*

- $\mu \circ \rho = \nu$.
- ρ is ramified only at places in T
- ρ is Fontaine-Laffaille at all places above p , and hence crystalline.

In particular, ρ is geometric. If we combine this with Proposition 3.8, we obtain Theorem 1.1.

Remark 3.13. Using the local deformation conditions for GL_n in [CHT08, §2.4.1] and [CHT08, §2.4.4], the same argument gives an identical result with $G = \mathrm{GL}_n$. But for $n > 2$ it is impossible to satisfy the oddness hypothesis. For GL_2 , this is a variant of [Ram02, Theorem 1b].

Remark 3.14. For other groups, the method will produce lifts provided appropriate local conditions exist. The deformation conditions we used are only available in full strength for symplectic and orthogonal groups. An alternative deformation condition above p is the ordinary deformation condition [Pat15, §4.1], available for any G . For ramified primes not above p , §6 provides a deformation condition assuming a certain nilpotent centralizer is smooth and $\bar{\rho}|_{\Gamma_v}$ is tamely ramified of the special type considered in §6.

4. REPRESENTATIVES FOR NILPOTENT ORBITS

As a first step on the road to defining the minimally ramified deformation condition, we study integral representatives of nilpotent orbits. Useful background about nilpotent orbits is collected in [Jan04].

4.1. Algebraically Closed Fields. In this subsection, we let k be algebraically closed of characteristic $p \geq 0$ and take G to be a connected reductive group over k with p good for G . (By convention, characteristic zero is good for all G .) Let $\mathfrak{g} = \text{Lie } G$. For a nilpotent $N \in \mathfrak{g}$, the orbit O_N of N under G can be defined as the locally closed (reduced) image of the orbit map through N . It is a smooth locally closed subscheme of \mathfrak{g} by the closed orbit lemma, and is called a *nilpotent orbit* (for G). As G is connected, this orbit is irreducible.

In good characteristic, the finite number of nilpotent orbits can be described by the Bala-Carter method as we review below. (More information can be found in [Jan04, §4], and a uniform proof without case-checking in small characteristic is due to Premet [Pre03].) It is important that the classification is combinatorial, in the sense that the nilpotent orbits can be described in a manner independent of p . To state it, we need to define some terminology. Let H be a connected reductive k -group with p good for H , and $\mathfrak{h} = \text{Lie } H$.

- For a parabolic $P \subset H$ with unipotent radical U , the *Richardson orbit* associated to P is the unique nilpotent orbit of H with dense intersection with $\text{Lie } U$. Its intersection with $\text{Lie } P$ is a single orbit under P .
- A parabolic subgroup $P \subset H$ with unipotent radical U is a *distinguished parabolic* if $\dim P/U = \dim U/U'$, where U' is the derived group of U .

Bala and Carter classified nilpotent orbits when the characteristic is good. One can check that if p is good for G , it will also be good for any Levi factor of a parabolic subgroup of G .

Fact 4.1. *If p is a good prime for G , the nilpotent orbits for G are in bijection with $G(k)$ -conjugacy classes of pairs (L, P) where L is a Levi factor of a parabolic subgroup of G and P is a distinguished parabolic of L . The nilpotent orbit for G associated to (L, P) is the unique one meeting $\text{Lie}(P)$ in the Richardson orbit of $P \subset L$.*

Example 4.2. For $G = \text{GL}_n$, conjugacy classes of parabolic subgroups of GL_n are indexed by partitions $n = n_1 + \dots + n_r$, with a Levi subgroup given by the product $\text{GL}_{n_1} \times \dots \times \text{GL}_{n_r}$ inside the associated block-upper-triangular parabolic subgroup. The only distinguished parabolic subgroups of GL_{n_i} are the Borel subgroups. Taking into account conjugation, we conclude that conjugacy classes of nilpotents are in bijection with partitions of n . A representative for each orbit is given by a block matrix in Jordan canonical form with eigenvalues zero and blocks of size n_1, \dots, n_r . This is worked out in detail in [Jan04, §4.1, 4.4, 4.8]. The nilpotent orbits are the same for SL_n [Jan04, §1.2].

The *Bala-Carter data* \mathcal{C} for G is the set of $G(k)$ -conjugacy classes of pairs (L, P) as above. The data is independent of k in the sense that it can be described completely in terms of the root datum of G as follows. All Levi subgroups L of a parabolic k -subgroup Q of G are a single $\mathcal{R}_{u,k}(Q)$ -orbit, so in Fact 4.1 we may restrict to one Q per $G(k)$ -conjugacy class and one L per Q . We may pick L so that it contains a (split) maximal torus T . After conjugation by $L(k)$, the distinguished parabolic subgroup $P \subset L$ may be assumed to contain T as well. But we know that parabolic subgroups Q of G containing T are in bijection with parabolic subsets of $\Phi(G, T)$ via $Q \mapsto \Phi(Q, T)$, so the possible Levi factors L of Q containing T are described just in terms of the root datum. Likewise, parabolic subgroups P of L containing T are in bijection with parabolic subsets of $\Phi(L, T)$. If we can characterize the condition that P is distinguished just in terms of the root data, this would mean that the Bala-Carter data can be described solely in terms of the root data and so is completely combinatorial.

We do so by constructing a grading on the Lie algebra of a parabolic P . Pick a Borel subgroup $B \subset G$ satisfying $T \subset B \subset P$. Let $\mathfrak{t} = \text{Lie}(T)$ and $\Delta \subset \Phi = \Phi(L, T)$ be the set of positive simple roots determined by B . There is a unique subset $I \subset \Delta$ such that $P = BW_I B$ where W_I is the subset of the Weyl group generated by reflections with respect to roots in I . Define a group homomorphism $f : \mathbf{Z}\Phi \subset \mathbf{Z}^\Delta \rightarrow \mathbf{Z}$ by specifying that on the basis Δ we have

$$f(\alpha) = \begin{cases} 2 & \alpha \in \Delta - I, \\ 0 & \alpha \in I. \end{cases}$$

This function gives a grading on $\mathfrak{l} = \text{Lie}(L)$:

$$\mathfrak{l}(i) = \bigoplus_{f(\alpha)=i} \mathfrak{l}_\alpha \quad \text{and} \quad \mathfrak{l}(0) = \left(\bigoplus_{f(\alpha)=0} \mathfrak{l}_\alpha \right) \oplus \mathfrak{t}$$

(sums indexed by $\alpha \in \Phi$). With respect to this grading,

$$\text{Lie } P = \bigoplus_{i \geq 0} \mathfrak{l}(i) \quad \text{and} \quad \text{Lie } U = \bigoplus_{i > 0} \mathfrak{l}(i).$$

The condition that P is distinguished is equivalent to the condition that

$$\dim \mathfrak{l}(0) = \dim \mathfrak{l}(2) + \dim Z_L$$

by [Car85, Corollary 5.8.3] as p is good for L . But this condition depends only on the root datum. Thus the Bala-Carter data for G can be described in a manner independent of the choice of algebraically closed field.

Definition 4.3. For $\sigma \in \mathcal{C}$, let $O_\sigma \subset \mathfrak{g}$ (or $O_{k,\sigma}$ if it is necessary to specify the field) be the corresponding nilpotent orbit.

From the classification of nilpotent orbits over algebraically closed fields, it is known that the corresponding nilpotent orbits in characteristic zero and characteristic p have the same dimension.

Example 4.4. Let $G = \text{GL}_n$ over an algebraically closed field k . For a partition $n = n_1 + n_2 + \dots + n_r$, define $d_i(\sigma)$ inductively by $d_0(\sigma) = 0$ and $d_i(\sigma) = d_{i-1}(\sigma) + \#\{j : n_j \geq i\}$. The orbit closure \overline{O}_σ consists of nilpotents $N \in \mathfrak{g}$ such that the $(n+1-d_i(\sigma)) \times (n+1-d_i(\sigma))$ minors of N^i vanish for all $i = 1, \dots, n$.

Remark 4.5. We consider connected reductive groups, but natural groups like O_n are disconnected. The nilpotent orbits of O_n and SO_n are different but closely related; as explained in [Jan04, §1.12], the only change is that certain pairs of orbits for SO_n may become a single O_n -orbit.

4.2. Integral Representatives. Now consider the case when G is a split reductive group scheme with connected fibers over a discrete valuation ring \mathcal{O} with $\mathfrak{g} = \text{Lie } G$. Let k be the residue field of characteristic $p > 0$. Assume p is very good for G_k . For $\sigma \in \mathcal{C}$, we seek elements

$$(4.1) \quad N_\sigma \in \mathfrak{g} \text{ such that } (N_\sigma)_k \in O_{\overline{k},\sigma} \text{ and } (N_\sigma)_K \in O_{\overline{K},\sigma}.$$

This makes precise the statement that N_K and N_k “lie in the same nilpotent orbit.”

In the case $G = \text{GL}_n$, we will see that the standard representatives in $G(\mathcal{O})$ for nilpotent orbits given in Jordan canonical form satisfy this condition. We will also explicitly describe such N_σ for symplectic and orthogonal groups.

Remark 4.6. More generally, one can obtain N_σ in terms of root data following [SS70, III.4.29]. We need the additional information provided by the concrete description in the symplectic and orthogonal cases to analyze the centralizer $Z_G(N)$ as an \mathcal{O} -scheme, so do not use this here.

For GL_n , it is easy to describe an N_σ as in (4.1) in terms of the root data via Jordan canonical form:

Example 4.7. When $G = \mathrm{GL}_n$, for the diagonal torus T and standard upper triangular Borel subgroup B , let e_i be the character extracting the i th diagonal entry of the torus. The positive simple roots in $\Phi(G, T)$ with respect to B are $\{\alpha_i = e_i - e_{i+1} : i = 1, \dots, n-1\}$. For a partition $n = n_1 + \dots + n_r$, the corresponding nilpotent orbit as in Example 4.2 corresponds to the standard upper triangular Borel subgroup B_L of $L = \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \dots \times \mathrm{GL}_{n_r}$. Now $\Phi(L, T)$ contains as its set Δ_L of simple roots with respect to B_L those $e_i - e_{i+1}$ with $i \neq n_1, n_1 + n_2, \dots, n_r$ (in other words, i and $i+1$ lie in the “same block”). We consider

$$N = \sum_{\alpha \in \Delta_L} N_\alpha$$

where N_α is a basis element for the root line \mathfrak{g}_α . Identifying \mathfrak{g}_k with $\mathrm{End}(k^n)$, N_k is the nilpotent matrix in Jordan canonical form whose blocks (in order) are of sizes n_1, n_2, \dots, n_r . Thus, N satisfies (4.1).

For symplectic and orthogonal groups, explicitly describing the nilpotents constructed from Bala-Carter data is more complicated. It turns out to be more convenient to construct and analyze representatives in these cases using a partition-based classification rather than via Bala-Carter data; this approach gives additional information about the centralizers that will be needed in §5.3. This is a minor extension of the classical results known over algebraically closed fields [Jan04, §1].

Let $G = \mathrm{Sp}_m$ with $m = 2n$, or $G = \mathrm{O}_m$ with $m = 2n$ or $m = 2n + 1$. We assume $n \geq 2$. Recall that Sp_m and O_m are defined using standard pairings on a free \mathcal{O} -module M of rank m . For $m = 2n$, the *standard alternating pairing* φ_{std} on \mathcal{O}^m is the one given by the block matrix

$$\begin{pmatrix} 0 & I'_n \\ -I'_n & 0 \end{pmatrix},$$

where I'_n denotes the anti-diagonal matrix with 1's on the diagonal. The *standard symmetric pairing* φ_{std} on \mathcal{O}^m is the one given by the matrix I'_m .

Remark 4.8. We chose to work with O_m instead of SO_m , as the classification is cleaner for O_m . The nilpotent orbits are almost the same for SO_m , except that certain nilpotent orbits of O_m (the ones where the partition contains only even parts) split into two SO_m -orbits [Jan04, Proposition 1.12] (conjugation by an element of O_m with determinant -1 carries one such orbit into the other).

Definition 4.9. Let σ denote a partition $m = m_1 + m_2 + \dots + m_r$ of m . It is *admissible* if

- every even m_i appears an even number of times when $G = \mathrm{O}_m$;
- every odd m_i appears an even number of times when $G = \mathrm{Sp}_m$.

The admissible partitions of m are in bijection with nilpotent orbits of Sp_m or O_m over an algebraically closed field [Jan04, Theorem 1.6]. The corresponding orbit is the intersection of $\mathfrak{g} \subset \mathfrak{gl}_m$ with the GL_m -orbit corresponding to that partition of m . Note that GL_m -orbit representatives in Jordan canonical form need not lie in \mathfrak{g} .

We will construct nilpotents together with a pairing, and then show how to relate the constructed pairing to the standard pairings used to define G . Let $\epsilon = 1$ in the case of O_m , and $\epsilon = -1$ in the case of Sp_m .

Definition 4.10. Let $d \geq 2$ be an integer. Define $M(d) = \mathcal{O}^d$, with basis v_1, \dots, v_d and a perfect pairing φ_d such that

$$\varphi_d(v_i, v_j) = \begin{cases} (-1)^i, & i + j = d + 1 \\ 0, & \text{otherwise} \end{cases}$$

(alternating for even d , symmetric for odd d). Define a nilpotent $N_d \in \mathrm{End}(M(d))$ by $N_d v_i = v_{i-1}$ for $1 < i \leq d$ and $N_d v_1 = 0$.

Similarly, define $M(d, d) = \mathcal{O}^{2d}$ with basis $v_1, \dots, v_d, v'_1, \dots, v'_d$ and a perfect ϵ -symmetric pairing $\varphi_{d,d}$ by extending

$$\varphi_{d,d}(v_i, v_j) = \varphi_{d,d}(v'_i, v'_j) = 0 \quad \text{and} \quad \varphi_{d,d}(v_i, v'_j) = \begin{cases} (-1)^i, & i + j = d + 1 \\ 0, & \text{otherwise} \end{cases}$$

Define a nilpotent $N_{d,d} \in \text{End}(M(d, d))$ by $N_{d,d}v_i = v_{i-1}$ and $N_{d,d}v'_i = v'_{i-1}$ for $1 < i \leq d$, and $N_{d,d}v_1 = N_{d,d}v'_1 = 0$.

It is straightforward to verify the pairings are perfect and that N_d and $N_{d,d}$ are skew with respect to the corresponding pairing. The pairing $\varphi_{d,d}$ can be symmetric or alternating.

Given an admissible partition $\sigma : m = m_1 + m_2 + \dots + m_r$, we will construct a free \mathcal{O} -module of rank m with an ϵ -symmetric perfect pairing and a nilpotent endomorphism respecting that pairing such that the Jordan block structure of nilpotent endomorphism on the geometric special fiber over $\text{Spec } \mathcal{O}$ is given by σ . Let $n_i(\sigma) = \#\{j : m_j = i\}$.

- If $G = \text{O}_m$ then $n_i(\sigma)$ is even for even i , so we can define

$$M_\sigma = \bigoplus_{i \text{ odd}} M(i)^{\oplus n_i(\sigma)} \oplus \bigoplus_{i \text{ even}} M(i, i)^{\oplus n_i(\sigma)/2}.$$

- If $G = \text{Sp}_m$ then $n_i(\sigma)$ is even for odd i , so we can define

$$M_\sigma = \bigoplus_{i \text{ odd}} M(i, i)^{\oplus n_i(\sigma)/2} \oplus \bigoplus_{i \text{ even}} M(i)^{\oplus n_i(\sigma)}.$$

Let φ_σ and N_σ denote the pairing and nilpotent endomorphism defined by the pairing and nilpotent endomorphism on each piece using Definition 4.10. In all cases, M_σ is a free \mathcal{O} -module of rank m . For each σ , let G_σ be the automorphism scheme $\underline{\text{Aut}}(M_\sigma, \varphi_\sigma)$, so for an algebraically closed field F over \mathcal{O} we have an isomorphism $(G_\sigma)_F \simeq G_F$ well-defined up to $G(F)$ -conjugation by using F -linear isomorphisms $(M_\sigma, \varphi_\sigma)_F \simeq (F^m, \varphi_{\text{std}})$.

Lemma 4.11. *For all admissible partitions of m , the specializations of the N_σ at geometric points ξ of $\text{Spec } \mathcal{O}$ constitute a set of representatives for the nilpotent orbits of G_ξ , and the specializations lie in the orbit corresponding to σ .*

Proof. The set of admissible partitions of m is in bijection with the set of nilpotent orbits over any algebraically closed field. The N_σ we constructed are integral versions of the representatives constructed in [Jan04, §1.7]. \square

Let e_1, e_2, \dots, e_m be the standard basis for \mathcal{O}^m . The elements e_i and e_{m+1-i} pair non-trivially under the standard pairing. When $m = 2n + 1$, e_{n+1} pairs non-trivially with itself under the standard pairing. We now relate the standard pairings to the pairings φ_σ .

Proposition 4.12. *Suppose that $\sqrt{-1}, \sqrt{2} \in \mathcal{O}^\times$. Then φ_σ is equivalent to the standard pairing over \mathcal{O} . There exists an \mathcal{O} -basis $\{v_i\}$ of \mathcal{O}^m with respect to which the pairing is given by φ_σ and N_σ satisfies the condition in (4.1) for $G = \text{Sp}_m$ or $G = \text{O}_m$.*

Proof. The standard pairings are very similar to φ_σ . In the case of Sp_m , each basis vector pairs trivially against all but one other basis vector, with which it pairs as ± 1 . So after reordering the basis, φ_σ is the standard pairing. The case of O_m is slightly more complicated. Let $\sigma : m = m_1 + m_2 + \dots + m_r$ be an admissible partition. The construction of M_σ and φ_σ gives a basis $\{v_{i,j}\}$ where $1 \leq i \leq r$ and $1 \leq j \leq m_i$. From the construction of φ_σ , we see that $v_{i,j}$ pairs trivially against all basis vectors except for v_{i, m_i+1-j} . So as long as $2j \neq m_i + 1$, we obtain a pair of basis vectors which are orthogonal to all others and which pair to ± 1 . For each odd m_i , the vector $v_{i, (m_i+1)/2}$ pairs non-trivially with itself. The standard pairing with respect to the basis e_i has such a vector only when $m = 2n + 1$ and then only for one e_i .

We must change the basis over \mathcal{O} so that φ_σ becomes the standard symmetric pairing. Let $v = v_{i,(m_i+1)/2}$ and $v' = v_{j,(m_j+1)/2}$ be two distinct vectors which pair non-trivially with themselves. In particular, $\varphi_\sigma(v, v) = (-1)^{(m_i+1)/2} := \eta$ and $\varphi_\sigma(v', v') = (-1)^{(m_j+1)/2} := \eta'$. Define

$$w = \frac{\sqrt{\eta}v - \sqrt{-\eta'}v'}{\sqrt{2}} \quad \text{and} \quad w' = \frac{\sqrt{\eta}v + \sqrt{-\eta'}v'}{\sqrt{2}}.$$

Then we see that $\varphi_\sigma(w, w) = 0 = \varphi_\sigma(w', w')$ and $\varphi_\sigma(w, w') = 1$. Making this change of variable over \mathcal{O} (which requires $\sqrt{-1}, \sqrt{2} \in \mathcal{O}^\times$), we have reduced the number of basis vectors which pair non-trivially with themselves by two, and produced a new pair of basis vectors orthogonal to the others and which pair to 1. By induction, we may therefore pick a basis v'_1, \dots, v'_m for which at most one basis vector pairs non-trivially with itself under φ_σ . After re-ordering, we may further assume that $\varphi_\sigma(v'_i, v'_j) = 0$ unless $i + j = m + 1$, in which case $\varphi_\sigma(v'_i, v'_j) = \pm 1$. Suppose $j = m + 1 - i$. If $i \neq j$, by scaling v'_i we may assume that $\varphi_\sigma(v'_i, v'_j) = 1$. If $i = j$, we already know that $\varphi_\sigma(v'_i, v'_j) = 1$. With respect to this basis, φ_σ is the standard pairing.

The last statement immediately follows from Lemma 4.11. \square

5. SMOOTHNESS OF CENTRALIZERS OF PURE NILPOTENTS

As before, let G be a split reductive group scheme with connected fibers over a discrete valuation ring \mathcal{O} with $\mathfrak{g} = \text{Lie } G$. Let k be the residue field of characteristic $p > 0$, and assume p is very good for G_k . For $N \in \mathfrak{g}$, denote the scheme-theoretic centralizer of N by $Z_G(N)$; it represents the functor

$$R \mapsto \{g \in G(R) : \text{Ad}_G(g)N_R = N_R\}$$

for \mathcal{O} -algebras R . We will study the centralizer $Z_G(N_\sigma)$ in more detail where $N_\sigma \in \mathfrak{g}$ is an element satisfying (4.1). In particular, this centralizer will be shown to be smooth when G is symplectic or orthogonal. We first review the known theory over fields, and then develop and apply a technique to deduce smoothness over \mathcal{O} (i.e. \mathcal{O} -flatness) from the known smoothness in the field case.

5.1. Centralizers over Fields. In this section, let k be an algebraically closed field, G be a connected reductive group over k , and N a nilpotent element of $\mathfrak{g} = \text{Lie } G$. As the formation of the scheme-theoretic centralizer commutes with base change, smoothness results for $Z_G(N)$ over k will imply such results over general fields (not necessarily algebraically closed).

The group scheme $Z_G(N)$ is the fiber over $0 \in \mathfrak{g}$ of the composition

$$G \xrightarrow{\text{Ad}_G} \text{GL}(\mathfrak{g}) \xrightarrow{T \mapsto TN - N} \mathfrak{g}.$$

Hence $\text{Lie } Z_G(N)$ is the kernel of

$$\mathfrak{g} \xrightarrow{\text{ad}_\mathfrak{g}} \text{End}(\mathfrak{g}) \xrightarrow{T \mapsto TN} \mathfrak{g}$$

which is the Lie algebra centralizer $\mathfrak{z}_\mathfrak{g}(N)$.

Remark 5.1. In references using the language of varieties rather than schemes (such as [Jan04]), $Z_G(N)$ is usually *defined* via its geometric points and hence is reduced and smooth, so the condition that the scheme $Z_G(N)$ is smooth becomes the condition that the variety $Z_G(N)$ has Lie algebra $\mathfrak{z}_\mathfrak{g}(N)$.

In a wide range of situations, all nilpotent centralizers are smooth. A direct calculation shows that this holds for $G = \text{GL}_n$ (see [Jan04, §2.3]), and a criterion of Richardson [Jan04, Theorem 2.5] can be applied to show:

Fact 5.2. *If G is an orthogonal or symplectic (similitude) group, any nilpotent centralizer is smooth over k .*

Remark 5.3. Suppose $Z_G(N)$ is smooth over k and p is good for G . The classification of nilpotent orbits is independent of p , as are their dimensions, so the dimension of $Z_G(N)$ is independent of p as well.

5.2. Checking Flatness over a Dedekind Base. We want to analyze smoothness of centralizers in the relative setting (especially over $\text{Spec } \mathcal{O}$). If $Z_G(N_\sigma) \rightarrow \text{Spec } \mathcal{O}$ is flat and the special and generic fibers are smooth then $Z_G(N)$ is smooth over \mathcal{O} . The following lemma gives a way to check that a morphism to a Dedekind scheme is flat.

Lemma 5.4. *Let $f : X \rightarrow S$ be finite type for a connected Dedekind scheme S . Then f is flat provided the following all hold:*

- (1) *for each $s \in S$, X_s is reduced and non-empty;*
- (2) *for each $s \in S$, X_s is equidimensional with dimension independent of s ;*
- (3) *there are sections $\{\sigma_i \in X(S)\}$ to f such that for every irreducible component of a fiber above a closed point, there is a section σ_i which meets the fiber only in that component.*

Remark 5.5. This lemma is a modification of [GY03, Proposition 6.1] to allow multiple irreducible components in the fibers.

Proof. It suffices to prove the result when $S = \text{Spec}(A)$ for A a discrete valuation ring with uniformizer π . Let X_η be the generic fiber and X_s the special fiber. Consider the schematic closure $\iota : X' \hookrightarrow X$ of the generic fiber. The scheme X' is flat over $\text{Spec}(A)$ since flatness is equivalent to being torsion-free over a discrete valuation ring, and there is an exact sequence

$$(5.1) \quad 0 \rightarrow J \rightarrow \mathcal{O}_X \rightarrow \iota_* \mathcal{O}_{X'} \rightarrow 0$$

where J is a coherent sheaf killed by a power of π . We will show that ι is an isomorphism by analyzing the special fiber.

First, we claim that the dimension of each irreducible component on the special fiber of X' is the same as the dimension of the equidimensional X_η . We will get this from flatness of X' . The generic fiber of X' is X_η , which is equidimensional and non-empty by hypothesis. Furthermore, X' is the union of the closures Z_i of the reduced irreducible components $X_{\eta,i}$ of X_η , and each Z_i is A -flat with integral η -fiber, hence integral. We just need to analyze the dimension of irreducible components of $(Z_i)_s$ when $(Z_i)_s \neq \emptyset$. Since Z_i is integral, we can apply [Mat89, Theorem 15.1, 15.5] to such Z_i to conclude that the dimension of each irreducible component of the special fiber of X' is the same as the dimension of the generic fiber.

Observe that the sections σ_i factor through the closed subscheme $X' \subset X$, as we can check this on the generic fiber since X' is A -flat. Thus X' meets every irreducible component of X_s away from the other irreducible components of X_s . We would have that $|X'_s| = |X_s|$ if X'_s is equidimensional of the same dimension as the equidimensional X_s . We have shown the dimension of any irreducible component in X'_s is the same dimension as the common dimension of irreducible components of the generic fiber X_η of X' . By hypothesis, the dimension of any irreducible component of the generic fiber of X is the same as the dimension of any irreducible component of the special fiber of X . Thus the dimension of any irreducible component of X'_s is the same as the dimension of each irreducible component of X_s , giving that $|X'_s| = |X_s|$. As X_s is reduced, this forces $\iota_s : X'_s \hookrightarrow X_s$ to be an isomorphism.

Now tensoring (5.1) with the residue field of A gives an exact sequence

$$0 \rightarrow J/\pi J \rightarrow \mathcal{O}_{X,s} \rightarrow \iota_* \mathcal{O}_{X',s} \rightarrow 0$$

because $\mathcal{O}_{X'}$ is A -flat. But $J/\pi J = 0$ as ι_s is an isomorphism. Hence $J = \pi J = \pi^2 J = \dots = \pi^n J = 0$ for n large, so $X = X'$ is flat over A . \square

Corollary 5.6. *In the situation of the lemma, if the fibers are also smooth then X is smooth.*

Proof. For a flat morphism of finite type between Noetherian schemes, smoothness of all fibers is equivalent to smoothness of the morphism. \square

5.3. Centralizers for Orthogonal and Symplectic Groups. To apply Corollary 5.6, we need information about the component group of centralizers of nilpotents. For GL_n over a field, all such centralizers are connected. For symplectic and orthogonal groups, there is an explicit description of $Z(N_\sigma)$ where N_σ is the nilpotent constructed in 4.2. We continue the notation of that section: G is Sp_m or O_m (with $m \geq 4$) over a discrete valuation ring \mathcal{O} with a residue field k of good characteristic $p \neq 2$.

Let $\sigma : m_1 + \dots + m_r$ be an admissible partition of m . We assume that \mathcal{O} is large enough so that Proposition 4.12 holds, and take $N := N_\sigma$. Then there exists elements $v_1, \dots, v_r \in M := \mathcal{O}^m$ such that

$$v_1, Nv_1, \dots, N^{m_1-1}v_1, v_2, Nv_2, \dots, N^{m_r-1}v_r$$

is a basis for M . Furthermore, $N^{d_i}v_i = 0$ for $i = 1, \dots, r$, and the pairing between basis elements is given by φ_σ . In particular, each v_i pairs non-trivially with only one other basis element $X^{d_i-1}v_{i^*}$, for some $i^* \in \{1, \dots, r\}$.

To understand the G -centralizer of N , we construct an associated grading of M as in [Jan04, §3.3, 3.4]. This is motivated by the Jacobson-Morosov theory of \mathfrak{sl}_2 -triples over a field of sufficiently large characteristic, but for symplectic and orthogonal groups it is constructed by hand in characteristic $p \neq 2$ below.

Remark 5.7. Every nilpotent $X \in \mathrm{End}(M_k)$ gives a filtration (and grading) of M_k defined by $\mathrm{Fil}^i = \ker(X^i)$. For GL_n , this is a nice filtration and is used in [CHT08] to define the minimally ramified deformation condition for GL_n . However, this filtration need not be isotropic with respect to the pairing, so we will construct a nicer grading associated to X .

Definition 5.8. Let $M(s)$ be the span of $N^j v_i$ for all i and j such that $s = 2j + 1 - d_i$. We set $M^{(s)} = \bigoplus_{t \geq s} M(t)$, and also define $L(s)$ to be the span of $\{v_i : v_i \in M(s)\}$.

Example 5.9. Take $G = \mathrm{O}_8$ and σ to be the admissible partition $1 + 2 + 2 + 3$. There are elements $v_1, v_2, v_3, v_4 \in M$ such that

$$v_1, v_2, Nv_2, v_3, Nv_3, v_4, Nv_4, N^2v_4$$

form a basis for M . Then $M(-2) = \mathrm{Span}_{\mathcal{O}}(v_4)$, $M(-1) = \mathrm{Span}_{\mathcal{O}}(v_2, v_3)$, $M(0) = \mathrm{Span}_{\mathcal{O}}(v_1, Nv_4)$, $M(1) = \mathrm{Span}_{\mathcal{O}}(Nv_2, Nv_3)$, and $M(2) = \mathrm{Span}_{\mathcal{O}}(N^2v_4)$. Furthermore, $L(0) = \mathrm{Span}_{\mathcal{O}}(v_1)$, $L(-1) = \mathrm{Span}_{\mathcal{O}}(v_2, v_3)$, and $L(-2) = \mathrm{Span}_{\mathcal{O}}(v_4)$. The nilpotent N raises the degree by 2, and $L(s)$ plays the role of a space of “lowest-weight vectors” inside $M(s)$ with respect to the operator N .

We now record some elementary properties of the preceding construction; all are routine to check, and the analogous proofs over a field may be found in [Jan04, §3.4]. We have that $M = \bigoplus_s M(s)$, and

$$v_i \in M(-(d_i - 1)), Nv_i \in M(-(d_i - 1) + 2), \dots, N^{d_i-1}v_i \in M(d_i - 1).$$

Furthermore, we know $N \cdot M(s) \subset M(s + 2)$ and $M(s) = N \cdot M(s - 2) \oplus L(s)$ for $s \leq 0$.

The dimension of $M(s)$ is $m_s(\sigma) := \{j : d_j - 1 \geq |s|\}$. The dimension of $L(s)$ equals $l_s(\sigma) := m_{s+1}(\sigma) - m_s(\sigma)$. Furthermore, the pairing φ interacts well with the grading: a computation with basis elements gives that $\varphi(M(s), M(t)) \neq 0$ implies $s + t = 0$.

The above grading on M corresponds to the one-parameter subgroup $\lambda : \mathbf{G}_m \rightarrow G$ for which the action of $t \in \mathbf{G}_m$ on $M(s)$ is given by scaling by t^s . The dynamic method (see [CGP15, §2.1]) associates to λ a parabolic subgroup $P_G(\lambda)$ with Levi $Z_G(\lambda)$. Define C_N and U_N to be the scheme-theoretic intersections

$$C_N = Z_G(N) \cap Z_G(\lambda) = \{g \in Z_G(N) : gM(i) = M(i) \text{ for all } i\}$$

$$U_N = Z_G(N) \cap U_G(\lambda) = \{g \in Z_G(N) : (g - 1)M^{(i)} \subset M^{(i+1)} \text{ for all } i\}.$$

Fact 5.10. *The group-scheme $Z_G(N)_k$ is a semi-direct product of $(C_N)_k$ and the smooth connected unipotent subgroup $(U_N)_k$. In particular, the connected components of $Z_G(N)_k$ are the same as the connected components of $(C_N)_k$.*

Remark 5.11. This is [Jan04, Proposition 3.12]. The existence of λ and this decomposition is not specific to symplectic and orthogonal groups [Jan04, Proposition 5.10].

We finally give a concrete description of C_N . We first define a pairing on $L(s)$. Recall that the space $L(s)$ of “lowest weight vectors” in $M(s)$ has basis $\{v_i : 1 - d_i = s\}$. We define a pairing on $L(s)$ by

$$\psi_s(v, w) = \varphi(v, N^{1-s}w).$$

A direct calculation shows that ψ_s is non-degenerate and that ψ_s is symmetric if $(-1)^{1-s} = -\epsilon$ and is alternating if $(-1)^{1-s} = \epsilon$ [Jan04, §3.7].

A point of C_N preserves the grading on M , and since it commutes with “raising operator” N its action on M is determined by its action on the space $L(s)$ of “lowest weight vectors” in $M(s)$, the following fact is no surprise.

Proposition 5.12. *There is an isomorphism of algebraic groups*

$$C_N \simeq \prod_{s \leq -1} \underline{\text{Aut}}(L(s), \psi_s)$$

The corresponding statement over a field is [Jan04, §3.8 Proposition 2, 3]: the proof is the same.

Example 5.13. Let $G = \text{Sp}_m$. Unraveling when ψ_s is symmetric or alternating, we see that

$$C_N \simeq \prod_{s \leq -1; s \text{ even}} \text{O}(L(s), \psi_s) \times \prod_{s \leq -1; s \text{ odd}} \text{Sp}(L(s), \psi_s).$$

The special fibers of the symplectic factors are connected, while the orthogonal factors have two connected components in the special fiber. Thus there are 2^t connected components, where t is the number of even s for which $L(s) \neq 0$. For each component, there is a section $g \in C_N(\mathcal{O})$ meeting that component corresponding to a choice of $\pm \text{Id} \in \text{O}(L(s), \psi_s)$ for each even s with $L(s) \neq 0$. The connected components of $Z_G(N)$ are the same as those for C_N by Fact 5.10.

Example 5.14. Let $G = \text{O}_m$. We likewise see that

$$C_N \simeq \prod_{s \leq -1; s \text{ odd}} \text{O}(L(s), \psi_s) \times \prod_{s \leq -1; s \text{ even}} \text{Sp}(L(s), \psi_s).$$

Again there are 2^t connected components of $Z_G(N)$, where t is the number of odd s for which $L(s) \neq 0$.

Now suppose that $G = \text{SO}_m$. The elements N we considered in this section are representatives for some of the nilpotent orbits of SO_m . The group C_N has the same structure as for $G = \text{O}_m$, except we require that the overall determinant be 1; this has 2^{t-1} connected components. Though SO_m has more nilpotent orbits than O_m , according to Remark 4.8 their representatives are conjugate by an element $\text{O}_m(k)$ with determinant -1 to the representatives constructed in Proposition 4.12. This shows that there are sections $g \in C_N(\mathcal{O})$ meeting each component.

Remark 5.15. Suppose q is a square in \mathcal{O}^\times . For use in the proof of Proposition 6.13, we need the existence of an element $\Phi \in G(\mathcal{O})$ such that $\text{ad}_G(\Phi)N_\sigma = qN_\sigma$. If $\alpha^2 = q$, taking $\Phi = \lambda(\alpha)$ would work: Φ would scale $N_\sigma^j v_i \in M(s)$ by α^s , and N_σ increases the degree by 2.

This Φ is a version for symplectic and orthogonal groups of the diagonal matrix denoted $\Phi(\sigma, a, q)$ whose diagonal entries are increasing powers of q used in [Tay08, §2.3]. There it is checked that $\text{ad}_G(\Phi(\sigma, a, q))N_\sigma = qN_\sigma$ where N_σ is the nilpotent representative in Jordan canonical form considered in Example 4.7 for the partition σ of m .

5.4. Smoothness of Centralizers. We now return the case when G is a split reductive group scheme with connected fibers over a discrete valuation ring \mathcal{O} with residue field k of *very good* characteristic $p > 0$. We assume that $\sqrt{-1}, \sqrt{2} \in \mathcal{O}$. Suppose we are given $N = N_\sigma \in \mathfrak{g} := \text{Lie } G$, an integral representative for the nilpotent orbit on geometric fibers corresponding to $\sigma \in \mathcal{C}$ as in (4.1): an element such that

$$N_k \in \mathcal{O}_{\bar{k}, \sigma} \text{ and } N_K \in \mathcal{O}_{\bar{K}, \sigma}.$$

Proposition 4.12 provides such N in symplectic and orthogonal cases as $\sqrt{-1}, \sqrt{2} \in \mathcal{O}^\times$. We wish to check that the $Z_G(N)$ is smooth over \mathcal{O} . An obviously necessary condition is that

$$(5.2) \quad Z_{G_K}(N_K) \text{ and } Z_{G_k}(N_k) \text{ are smooth of the same dimension.}$$

Remark 5.16. Some assumption on N is essential. Otherwise $N_{\bar{K}}$ and $N_{\bar{k}}$ can lie in different nilpotent orbits (in terms of the combinatorial characteristic-free classification of geometric orbits), and so $Z_{G_K}(N_K)$ and $Z_{G_k}(N_k)$ could have different dimensions, in which case $Z_G(N)$ cannot be \mathcal{O} -flat. An example of this is the element N_2 in Example 1.6.

We define

$$A(N) = (Z_{G_k}(N_k)/Z_{G_k}(N_k)^\circ)(k) = Z_{G_k}(N_k)(k)/Z_{G_k}(N_k)^\circ(k),$$

and study when the following holds:

$$(5.3) \quad \text{each element of } A(N) \text{ arises from some } s \in Z_G(N)(\mathcal{O}).$$

This information is necessary to apply Corollary 5.6. We are free to make a local flat extension of \mathcal{O} , as it suffices to check flatness after such an extension. In particular, it suffices to check (5.3) when \mathcal{O} is Henselian and k is algebraically closed. Examples 5.13 and 5.14 give such sections when $G = \text{Sp}_m$ or $G = \text{SO}_m$. We will use these cases to get a result for similitude groups.

Let $\pi : \widetilde{G}' \rightarrow G'$ be the simply connected central cover of the derived group G' over \mathcal{O} . As p is very good, \widetilde{G}' and G' have isomorphic Lie algebras via π and $\text{Lie } G'$ is a direct factor of $\text{Lie } G$ with complement $\text{Lie}(Z_G)$, so we may abuse notation and view N as an element of all of these Lie algebras over \mathcal{O} .

Let S be a (split) maximal central torus in G . Consider the isogeny $S \times \widetilde{G}' \rightarrow G$. As S acts trivially on N , we see that $S \times Z_{\widetilde{G}'}(N)$ is the preimage of $Z_{G'}(N)$ under this isogeny. As p is very good for G , we obtain finite étale surjections

$$Z_{\widetilde{G}'}(N) \rightarrow Z_{G'}(N) \quad \text{and} \quad S \times Z_{\widetilde{G}'}(N) \rightarrow Z_G(N)$$

over \mathcal{O} .

Lemma 5.17. *The condition (5.3) holds for \widetilde{G}' if and only if (5.3) holds for G .*

Proof. Assume \widetilde{G}' satisfies (5.3). Pick a connected component C of $Z_{G_k}(N_k)$. The preimage of C under $S \times Z_{\widetilde{G}'}(N) \rightarrow Z_G(N)$ is a union of k -fiber components of the form $S_k \times C'$ where C' is a connected component of $Z_{\widetilde{G}'_k}(N_k)$. By assumption, there exists $s \in Z_{\widetilde{G}'}(N)(\mathcal{O})$ meeting any such C' . The image of $(1, s)$ is a point of $Z_G(N)(\mathcal{O})$ meeting C .

Conversely, assume G satisfies (5.3). Pick a connected component C' of $Z_{\widetilde{G}'_k}(N_k)$. Under $S \times Z_{\widetilde{G}'}(N) \rightarrow Z_G(N)$, $S_k \times C'$ maps onto a connected component C of $Z_{G_k}(N_k)$. By assumption, there exists $s \in Z_G(N)(\mathcal{O})$ such that $s_k \in C$. As k is algebraically closed, there is $s'_k \in (S \times Z_{\widetilde{G}'}(N))(k)$ lifting s_k and lying in C' . As $S \times Z_{\widetilde{G}'}(N) \rightarrow Z_G(N)$ is a finite étale cover and \mathcal{O} is Henselian, there exists $s' \in (S \times Z_{\widetilde{G}'}(N))(\mathcal{O})$ lifting s and reducing to s'_k . \square

For example, this lets us pass between Sp_{2n} and GSp_{2n} by way of the projective symplectic group.

Proposition 5.18. *For G a symplectic or orthogonal similitude group, and $N = N_\sigma \in \mathfrak{g}$ the element satisfying (4.1) given by Proposition 4.12 for an admissible partition σ , the centralizer $Z_G(N)$ is smooth over \mathcal{O} .*

Proof. By Fact 5.2, $Z_{G_k}(N_k)$ and $Z_{G_K}(N_K)$ are smooth. By the classification of nilpotent orbits over algebraically closed fields, the dimension of the orbit only depends on the combinatorial classification for the orbit in very good characteristic and in characteristic 0, so these fibers are equidimensional of the same dimension. By Corollary 5.6, it suffices to find $s \in Z_G(N)(\mathcal{O})$ meeting any connected component of $Z_{G_k}(N_k)$.

Using Lemma 5.17, we reduce checking (5.3) to the cases of Sp_m and SO_m , covered by Examples 5.13 and 5.14. \square

Remark 5.19. Consider the nilpotent orbits of GL_n as in Example 4.2, with representative N given in Example 4.7. As $Z_{G_k}(N_k)$ is connected [Jan04, Proposition 3.10], the identity section shows (5.3) holds. This shows $Z_G(N)$ is smooth.

Remark 5.20. It is not hard to extend the above argument to work for groups such that all the irreducible factors of the root system are of classical type. For the exceptional groups, one could find N as in Remark 4.6 and attempt to check (5.3) holds by hand (there are finitely many cases). A conceptual approach would be preferable.

Remark 5.21. McNinch analyzes the centralizer of an “equidimensional nilpotent” in [McN08]. An *equidimensional nilpotent* is an element $N \in \mathfrak{g}$ such that N_K is nilpotent and the dimension of the special and generic fibers of $Z_G(N)$ are the same. [McN08, §5.2] claims that such $Z_G(N)$ are \mathcal{O} -smooth because the fibers are smooth of the same dimension. This deduction is incorrect: it relies on [McN08, 2.3.2] which uses the wrong definition of an equidimensional morphism and thereby incorrectly applies [SGA1, Exp. II, Prop 2.3].

According to [SGA1, Exp. II, Prop 2.3] (or [EGAIV₃, §13.3, 14.4.6, 15.2.3]), for a Noetherian scheme Y , a morphism $f : X \rightarrow Y$ locally of finite type, and points $x \in X$ and $y = f(x)$ with \mathcal{O}_y normal, f is smooth at x if and only if f is equidimensional at x and $f^{-1}(y)$ is smooth over $k(y)$ at x . But by definition in [EGAIV₃, 13.3.2], an *equidimensional* morphism is more than just a morphism all of whose fibers are of the same dimension (the condition checked in [McN08, 2.3.2]): a locally finite type morphism f is called equidimensional of dimension d at $x \in X$ when there exists an open neighborhood U of x such that for every irreducible component Z of U through x , $f(Z)$ is dense in some irreducible component of Y containing y and for all $x' \in U$ the fiber $f^{-1}(f(x')) \cap U$ has all irreducible components of dimension d .

This is much stronger than the fibers simply being of the same dimension. To see the importance of the extra conditions, consider a discrete valuation ring \mathcal{O} with field of fractions K and residue field k , and the morphism from X , the disjoint union of $\mathrm{Spec} K$ and $\mathrm{Spec} k$, to $Y = \mathrm{Spec} \mathcal{O}$. The fibers are of the same dimension (zero) and smooth but the morphism is not flat. This morphism is also not equidimensional at $\mathrm{Spec} k$: the only irreducible component of X containing $\mathrm{Spec} k$ is the point itself, with image the closed point of $\mathrm{Spec} \mathcal{O}$. This is not dense in $\mathrm{Spec} \mathcal{O}$, the only irreducible component of the only open set containing the closed point of $\mathrm{Spec} \mathcal{O}$.

The smoothness of centralizers of an equidimensional pure nilpotent is important to proving the main results of [McN08]. In particular, the results in §6 and §7 in [McN08] crucially rely on the smoothness of the centralizers of such nilpotents, leaving a gap in the proof of Theorem B in [McN08] concerning the component group of centralizers. The method we have discussed here reverses this, understanding the geometric component group well enough to *produce* sufficiently many \mathcal{O} -valued points in order to deduce smoothness of the centralizer in classical cases in very good characteristic via Lemma 5.4.

6. MINIMALLY RAMIFIED DEFORMATIONS: TAME CASE

In this section, we will generalize the tamely ramified case of the minimally ramified deformation condition introduced in [CHT08, §2.4.4] for GL_n to symplectic and orthogonal groups. We also explain why another more immediate notion based on parabolic subgroups, giving the same deformation condition for GL_n , is *not liftable* in general (even for GSp_4). We begin by defining the notion of a pure nilpotent lift and then define and study the deformation condition.

6.1. Pure Nilpotent Lifts. As before, let \mathcal{O} be a discrete valuation ring with residue field k of characteristic $p > 0$, and let G be a split reductive group scheme over \mathcal{O} (with connected fibers) such that p is very good for G . Let $\mathfrak{g} = \mathrm{Lie} G$. For a nilpotent element $\overline{N} \in \mathfrak{g}_k$ of type $\sigma \in \mathcal{C}$, we will define the notion of a *pure nilpotent* lift of \overline{N} in \mathfrak{g} and study the space of such lifts, assuming there exists $N_\sigma \in \mathfrak{g}$ lifting \overline{N} such that $(N_\sigma)_{\overline{k}} \in \mathcal{O}_{\overline{k}, \sigma}$ and such that $Z_G(N_\sigma)$ is smooth over \mathcal{O} .

Remark 6.1. For GL_n and for orthogonal and symplectic (similitude) groups, Remark 5.19 and Proposition 5.18 show that for any nilpotent $\overline{N} \in \mathfrak{g}_k$, there exists $N'_\sigma \in \mathfrak{g}$ such that $(N'_\sigma)_{\overline{k}} \in \mathcal{O}_{\overline{k}, \sigma}$, $Z_G(N'_\sigma)$ is \mathcal{O} -smooth, and such that $(N'_\sigma)_k$ and \overline{N} are $G(\overline{k})$ -conjugate. Thus $(N'_\sigma)_k$ and \overline{N} are conjugate by $\overline{g} \in G(k')$ for some finite extension k'/k . Lift \overline{g} to an element $g \in G(\mathcal{O}')$ for a Henselian discrete valuation ring local over \mathcal{O} and having residue field k' . The element $N_\sigma := gN'_\sigma g^{-1} \in \mathfrak{g}_{\mathcal{O}'}$ reduces to $\overline{N}_{k'}$ and has the required properties. So the above hypothesis is satisfied after a finite flat local extension of \mathcal{O} .

Definition 6.2. Define the functor $\mathrm{Nil}_{\overline{N}} : \mathcal{C}_{\mathcal{O}} \rightarrow \mathrm{Sets}$ by

$$\mathrm{Nil}_{\overline{N}}(R) = \{N \in \mathfrak{g}_R \mid \mathrm{Ad}_G(g)(N_\sigma) = N \text{ for some } g \in \widehat{G}(R)\}.$$

Call these $N \in \mathrm{Nil}_{\overline{N}}(R)$ the *pure nilpotents* lifting \overline{N} .

This is obviously a subfunctor of the formal neighborhood of \overline{N} in the affine space \mathfrak{g} over \mathcal{O} attached to \mathfrak{g} . The key to analyzing $\mathrm{Nil}_{\overline{N}}$ is that $Z_{G_R}(N)$ is smooth over R since $Z_G(\overline{N}_\sigma)$ is \mathcal{O} -smooth and $(N_\sigma)_R$ is in the G -orbit of N . To ease notation below, we shall write gNg^{-1} rather than $\mathrm{Ad}_G(g)(N)$ for $g \in \widehat{G}(R)$.

Lemma 6.3. *The functor $\mathrm{Nil}_{\overline{N}}$ is pro-representable.*

Proof. We will use Schlessinger's criterion to check pro-representability. As $\mathrm{Nil}_{\overline{N}}$ is a subfunctor of the formal neighborhood of the scheme \mathfrak{g} at \overline{N} , the only condition to check is the analogue of Definition 2.5(2): given a Cartesian diagram in $\mathcal{C}_{\mathcal{O}}$

$$\begin{array}{ccc} R_1 \times_{R_0} R_2 & \xrightarrow{\pi_2} & R_2 \\ \downarrow \pi_1 & & \downarrow \\ R_1 & \longrightarrow & R_0 \end{array}$$

and $N_i \in \mathrm{Nil}_{\overline{N}}(R_i)$ such that N_1 and N_2 reduce to N_0 , we want to check that $N_1 \times_{R_0} N_2 \in \mathrm{Nil}_{\overline{N}}(R_1 \times_{R_0} R_2)$. By definition, there exists $g_1 \in \widehat{G}(R_1)$ and $g_2 \in \widehat{G}(R_2)$ such that $N_1 = g_1 N_\sigma g_1^{-1}$ and $N_2 = g_2 N_\sigma g_2^{-1}$. Consider the element $g_1 g_2^{-1} \in \widehat{G}(R_0)$. Observe that

$$g_1 g_2^{-1} N_\sigma g_2 g_1^{-1} = g_1 N_\sigma g_1^{-1} = N_\sigma \in \mathfrak{g}_{R_0}.$$

In particular, $g_1 g_2^{-1} \in Z_G(N_\sigma)(R_0)$. The extension $R_2 \rightarrow R_0$ has nilpotent kernel, so as $Z_G(N_\sigma)$ is smooth over \mathcal{O} there exists $h \in Z_G(N_\sigma)(R_2)$ lifting $g_1 g_2^{-1}$. The element

$$(g_1, hg_2) \in R_1 \times_{R_0} R_2$$

conjugates $N_1 \times_{R_0} N_2$ to N_σ . Hence $N_1 \times_{R_0} N_2 \in \mathrm{Nil}_{\overline{N}}(R_1 \times_{R_0} R_2)$. \square

Lemma 6.4. *The functor $\text{Nil}_{\overline{N}}$ is formally smooth, in the sense that for a small surjection $R_2 \rightarrow R_1$ of coefficient \mathcal{O} -algebras the map*

$$\text{Nil}_{\overline{N}}(R_2) \rightarrow \text{Nil}_{\overline{N}}(R_1)$$

is surjective. Moreover, $\text{Nil}_{\overline{N}}$ has relative dimension $\dim G_k - \dim Z_{G_k}(N_k)$ over \mathcal{O} .

Proof. Given $N \in \text{Nil}_{\overline{N}}(R_1)$, there exists $g \in \widehat{G}(R_1)$ such that $gNg^{-1} = N_\sigma$. As G is smooth over \mathcal{O} , we may find $g' \in \widehat{G}(R_2)$ lifting g . Then $(g')^{-1}N_\sigma g'$ is a lift of N to R_2 . From its definition, the tangent space to $\text{Nil}_{\overline{N}}$ is $\mathfrak{g}_k/\mathfrak{z}_{\mathfrak{g}}(N_k)$, so the formally smooth $\text{Nil}_{\overline{N}}$ has relative dimension $\dim G_k - \dim Z_{G_k}(N_k)$ since $Z_G(N)$ is \mathcal{O} -smooth. \square

Now suppose that A is a complete local Noetherian \mathcal{O} -algebra with residue field k .

Lemma 6.5. *The inverse limit $\varprojlim \text{Nil}_{\overline{N}}(A/\mathfrak{m}_A^n)$ equals $\{N \in \mathfrak{g}_A : N = gN_\sigma g^{-1} \text{ for some } g \in G(A)\}$.*

Proof. It is immediate that the second is a subset of the first. On the other hand, suppose we had compatible elements $N_i \in \text{Nil}_{\overline{N}}(A/\mathfrak{m}_A^i)$ such that N_i is $\widehat{G}(A/\mathfrak{m}_A^i)$ -conjugate to N_σ .

By induction, we will show there exists $g_i \in \widehat{G}(A/\mathfrak{m}_A^i)$ such that $N_i = g_i N_\sigma g_i^{-1}$ and g_i reduces to g_{i-1} . The base case $i = 1$ is just the assertion that $(N_\sigma)_k$ equals N_1 . Given $g_i \in \widehat{G}(A/\mathfrak{m}_A^i)$, we know there is some element $g'_{i+1} \in \widehat{G}(A/\mathfrak{m}_A^{i+1})$ such that $N_i = g'_{i+1} N_\sigma (g'_{i+1})^{-1}$. The element $(g'_{i+1})^{-1} g_i$ lies in $Z_G(N_\sigma)(A/\mathfrak{m}_A^i)$. As $Z_G(N_\sigma)$ is smooth over \mathcal{O} , we may lift to produce an element $\tilde{g} \in Z_G(N_\sigma)(A/\mathfrak{m}_A^{i+1})$ for which $N_i = g'_{i+1} \tilde{g} N_\sigma (g'_{i+1} \tilde{g})^{-1}$ and such that $g'_{i+1} \tilde{g}$ reduces to $g_i \in \widehat{G}(A/\mathfrak{m}_A^i)$. This completes the induction.

Finally let $g \in \widehat{G}(A)$ be the limit of the g_i and observe $gN_\sigma g^{-1}$ is the limit of the N_i . \square

Remark 6.6. If we had defined $\text{Nil}_{\overline{N}}$ on the larger category $\widehat{\mathcal{C}}_{\mathcal{O}}$ in the obvious way, Lemma 6.5 would say that $\text{Nil}_{\overline{N}}$ is continuous.

Remark 6.7. One can define a scheme-theoretic “nilpotent cone” over \mathcal{O} as the vanishing locus of the ideal of non-constant homogeneous G -invariant polynomials on \mathfrak{g} . The arguments in this section could be rephrased as constructing a formal scheme of pure nilpotents inside the formal neighborhood of \overline{N} in \mathfrak{g} . A natural question is whether there is a broader notion of pure nilpotents that gives a locally closed subscheme of the scheme-theoretic nilpotent cone. For instance, for $N, N' \in \mathfrak{g}$, if their images in \mathfrak{g}_K and \mathfrak{g}_k are nilpotent in orbits with the same combinatorial parameters, are N and N' conjugate under G over a discrete valuation ring local over \mathcal{O} ?

When $G = \text{GL}_n$, this has been explored by Taylor in the course of constructing local deformation conditions [Tay08, Lemma 2.5]. The method uses the explicit description of the orbit closures in Example 4.4 to define an analogue of the orbit closures over \mathcal{O} . It would be interesting to find a way to do the same for a general split connected reductive group.

6.2. Passing between Unipotents and Nilpotents. We now specialize to the case that G is either GSp_m or GO_m (or GL_m to recover the results of [CHT08, §2.4.4]) over the ring of integers \mathcal{O} in a p -adic field with residue field k of characteristic $p > 0$ with $m \geq 4$. As always, we assume that p is very good for G_k (i.e. $p \neq 2$). Let $\mathfrak{g} = \text{Lie}(G)$.

As in §6.1, we work with a pure nilpotent $N_\sigma \in \mathfrak{g}$ for which $Z_G(N_\sigma)$ is \mathcal{O} -smooth, $(N_\sigma)_{\overline{K}} \in \mathcal{O}_{\overline{K}, \sigma}$, and $(N_\sigma)_{\overline{k}} \in \mathcal{O}_{\overline{k}, \sigma}$. Define $\overline{N} := (N_\sigma)_k$. We studied deformations of \overline{N} in §6.1, but will ultimately want to analyze deformations of Galois representations which take on unipotent values at certain elements of a local Galois group. Thus, we need a way to pass between unipotent and nilpotent elements. For classical groups, we can use a truncated version of the exponential and logarithm maps:

Fact 6.8. *Suppose that $p \geq m$ and that R is an \mathcal{O} -algebra. If $A \in \text{Mat}_m(R)$ has characteristic polynomial x^m then*

$$\exp(A) := 1 + A + A^2/2 + \dots + A^{m-1}/(m-1)!$$

has characteristic polynomial $(x-1)^m$. If $B \in \text{Mat}_m(R)$ has characteristic polynomial $(x-1)^m$ then

$$\log(B) := (B-1) - (B-1)^2/2 + \dots + (-1)^m(B-1)^{m-1}/(m-1)$$

has characteristic polynomial x^m . Furthermore for $C \in \text{GL}_m(R)$ and an integer q , we have

- $\exp(CAC^{-1}) = C \exp(A) C^{-1}$
- $\log(CBC^{-1}) = C \log(B) C^{-1}$
- $\log(\exp(A)) = A$
- $\exp(\log(B)) = B$
- $\exp(qA) = \exp(A)^q$
- $\log(B^q) = q \log(B)$

This is [Tay08, Lemma 2.4]. The key idea is that because all the higher powers of A and $B-1$ vanish and all of the denominators appearing are invertible as $p \geq m$, we can deduce these facts from results about the exponential and logarithm in characteristic zero.

Suppose J is the matrix for a perfect symmetric or alternating pairing over R .

Corollary 6.9. *For A and B as in Fact 6.8 with $\exp(A) = B$, $A^T J + JA = 0$ if and only if $B^T JB = J$.*

Proof. Directly from the definitions we see that $\exp(A^T) = \exp(A)^T$. Observe that $\exp(JAJ^{-1}) = JB J^{-1}$ and $\exp(-A^T) = (B^T)^{-1}$. Thus $JAJ^{-1} = -A^T$ if and only if $(B^T)^{-1} = JB J^{-1}$. \square

We shall use this exponential map to convert pure nilpotents into unipotent elements. Let R be a coefficient ring over \mathcal{O} . By Definition 6.2, any pure nilpotent $N \in \text{Nil}_{\overline{N}}(R)$ is $G(R)$ -conjugate to N_σ , so it has characteristic polynomial x^m . Denoting the derived group of G by G' , any nilpotent element of \mathfrak{g} lies in $(\mathfrak{g}') = (\text{Lie } G')$, so $NJ + JN = 0$ (and not just $NJ + JN = \lambda J$ for some $\lambda \in \mathcal{O}$). Thus, Corollary 6.9 shows that $\exp(N) \in G(R)$. This gives an exponential map

$$(6.1) \quad \exp : \text{Nil}_{\overline{N}} \rightarrow G$$

such that for $g \in \widehat{G}(R)$, $N \in \text{Nil}_{\overline{N}}(R)$, and $q \in \mathbf{Z}$ we have $\exp(qN) = \exp(N)^q$ and $g \exp(N) g^{-1} = \exp(\text{Ad}_G(g)N)$.

Remark 6.10. This is a realization over \mathcal{O} of a special case of the Springer isomorphism identifying the nilpotent and unipotent varieties in very good characteristic. For later purposes, we will need that the identification is compatible with the multiplication in the sense that $\exp(qA) = \exp(A)^q$. In the case of GL_m , a Springer isomorphism that works in any characteristic is given by $X \rightarrow 1 + X$ for nilpotent X , but this is not compatible with multiplication.

6.3. Minimally Ramified Deformations. As before, G is GSp_m , GO_m or GL_m over the ring of integers \mathcal{O} of a p -adic field with residue field k with $p \geq m$ (and $m \geq 4$ in the orthogonal and symplectic cases). Let L be a finite extension of \mathbf{Q}_ℓ (with $\ell \neq p$), and denote its absolute Galois group by Γ_L . Consider a representation $\overline{\rho} : \Gamma_L \rightarrow G(k)$. We wish to define a (large) smooth deformation condition for $\overline{\rho}$ generalizing the minimally ramified deformation condition for GL_n defined in [CHT08, §2.4.4]. In this section we do so for a special class of tamely ramified representations. This requires making an étale local extension of \mathcal{O} , which will be harmless for our purposes.

Recall that Γ_L^t , the Galois group of the maximal tamely ramified extension of L , is isomorphic to the semi-direct product

$$\widehat{\mathbf{Z}} \ltimes \prod_{p' \neq \ell} \mathbf{Z}_{p'}$$

where $\widehat{\mathbf{Z}}$ is generated by a Frobenius ϕ and the conjugation action by ϕ on the direct product is given by the cyclotomic character. We consider representations of Γ_L^t which factor through the quotient $\widehat{\mathbf{Z}} \rtimes \mathbf{Z}_p$. Picking a topological generator τ for \mathbf{Z}_p , the action is explicitly given by

$$\phi\tau\phi^{-1} = q\tau$$

where q is the size of the residue field of L . Note q is a power of ℓ , so it is relatively prime to p . This leads us to study representations of the group $T_q := \widehat{\mathbf{Z}} \rtimes \mathbf{Z}_p$.

Let $\bar{\rho} : T_q \rightarrow G(k)$ be such a representation. We first claim that $\bar{\rho}(\tau) \in G(k)$ is unipotent. This element decomposes as a commuting product of semisimple and unipotent elements of $G(k)$. The order of a semisimple element in $G(k)$ is prime to p , while by continuity there is an $r \geq 0$ such that $\tau^{p^r} \in \ker(\bar{\rho})$. Thus $\bar{\rho}(\tau)$ is unipotent.

Informally, a deformation $\rho : T_q \rightarrow G(R)$ will be minimally ramified if $\rho(\tau)$ lies in the “same” unipotent orbit as $\bar{\rho}(\tau)$. To make this meaningful over an infinitesimal thickening of k , we shall use the notion of pure nilpotents as in Definition 6.2 since unipotence and unipotent orbits are not good notions when not over a field. As $\bar{N} := \log(\bar{\rho}(\tau))$ is nilpotent, by Remark 6.1 after making an étale local extension of \mathcal{O} we may assume that there exists a pure nilpotent $N_\sigma \in \mathfrak{g}$ lifting \bar{N} for which $Z_G(N_\sigma)$ is smooth. Making a further extension if necessary, we may also assume that the unit $q \in \mathcal{O}^\times$ is a square. We obtain an exponential map $\exp : \text{Nil}_{\bar{N}} \rightarrow G$ as in (6.1).

Definition 6.11. For a coefficient ring R over \mathcal{O} , a continuous lift $\rho : T_q \rightarrow G(R)$ of $\bar{\rho}$ is *minimally ramified* if $\rho(\tau) = \exp(N)$ for some $N \in \text{Nil}_{\bar{N}}(R)$.

Example 6.12. Take $G = \text{GL}_n$. Then $X \mapsto 1_n + X$ gives an identification of nilpotents and unipotents. Up to conjugacy, over algebraically closed fields parabolic subgroups correspond to partitions of n and every nilpotent orbit is the Richardson orbit of such a parabolic. Let $\bar{\rho}(\tau) - 1_n =: \bar{N}$ correspond to the partition $\sigma = n_1 + n_2 + \dots + n_r$. By Example 4.7, the lift N_σ of \bar{N} is conjugate to a block nilpotent matrix with blocks of size n_1, n_2, \dots, n_r . The points $N \in \text{Nil}_{\bar{N}}(R)$ are the $\widehat{G}(R)$ -conjugates of N_σ . It is clear (since $p > n$) that if $\rho(\tau) \in \text{Nil}_{\bar{N}}(R)$ then

$$(6.2) \quad \ker(\rho(\tau) - 1_n)^i \otimes_R k \rightarrow \ker(\bar{\rho}(\tau) - 1_n)^i$$

is an isomorphism for all i . Conversely, repeated applications of [CHT08, Lemma 2.4.15] show that any $\rho(\tau)$ satisfying this collection of isomorphism conditions is $\widehat{G}(R)$ -conjugate to N_σ . Thus the minimally ramified deformation condition for GL_n defined in [CHT08] agrees with our definition. Note that the identification $X \mapsto 1_n + X$ does not satisfy $qX \rightarrow (1 + X)^q$, so it will not work in our argument. The proof of [CHT08, Lemma 2.4.19] uses a different method for which this non-homomorphic identification suffices.

Proposition 6.13. *Under our assumptions on G , the local deformation ring $R_{\bar{\rho}}^{\text{m.r.}, \square}$ is formally smooth over \mathcal{O} of relative dimension $\dim G_k$.*

Proof. Let $\bar{\Phi} = \bar{\rho}(\phi) \in G(k)$ and let $\widehat{G}_{\bar{\Phi}}$ be the formal completion of G at $\bar{\Phi}$. Using the relation

$$\bar{\rho}(\phi)\bar{\rho}(\tau)\bar{\rho}(\phi)^{-1} = \bar{\rho}(\tau)^q,$$

we deduce that $\bar{\Phi}\bar{N}\bar{\Phi}^{-1} = q\bar{N}$. Therefore we study the functor $M_{\bar{N}}$ on $\widehat{\mathcal{C}}_{\mathcal{O}}$ defined by

$$M_{\bar{N}}(R) = \{(\Phi, N) : N \in \text{Nil}_{\bar{N}}(R), \Phi \in \widehat{G}_{\bar{\Phi}}(R), \Phi N \Phi^{-1} = qN\} \subset \text{Nil}_{\bar{N}}(R) \times \widehat{G}_{\bar{\Phi}}(R).$$

Any such lift (Φ, N) to a coefficient ring R determines a homomorphism $T_q \rightarrow G(R)$ lifting $\bar{\rho}$ via $\phi \mapsto \Phi$ and $\tau \mapsto \exp(N)$: it is continuous because $\exp(\bar{N})$ is unipotent. We will analyze $M_{\bar{N}}$ through the composition

$$M_{\bar{N}} \rightarrow \text{Nil}_{\bar{N}} \rightarrow \text{Spf } \mathcal{O}.$$

First, observe that $M_{\bar{N}} \rightarrow \text{Nil}_{\bar{N}}$ is relatively representable as “ $\Phi N = qN\Phi$ ” is a formal closed condition on points Φ of $(\widehat{G}_{\bar{\Phi}})_R$ for each $N \in \text{Nil}_{\bar{N}}(R)$.

From Lemma 6.4, we know that $\mathrm{Nil}_{\overline{N}}$ is formally smooth over \mathcal{O} , and the universal nilpotent is $gN_{\sigma}g^{-1}$ for some $g \in \widehat{G}(\mathrm{Nil}_{\overline{N}})$. To check formal smoothness of the map $M_{\overline{N}} \rightarrow \mathrm{Nil}_{\overline{N}}$, it therefore suffices to check the formal smoothness of the fiber of $M_{\overline{N}}$ over the \mathcal{O} -point N_{σ} of $\mathrm{Nil}_{\overline{N}}$.

We have written down $\Phi_{\sigma} \in G(\mathcal{O})$ satisfying $\Phi_{\sigma}N_{\sigma}\Phi_{\sigma}^{-1} = qN_{\sigma}$ in Remark 5.15. Observe that $\overline{\Phi}\overline{\Phi}_{\sigma}^{-1} \in Z_G(N_{\sigma})(k)$. By smoothness, we may lift $\overline{\Phi}\overline{\Phi}_{\sigma}^{-1}$ to an element $s \in Z_G(N_{\sigma})(\mathcal{O})$. Then $s\Phi_{\sigma}$ reduces to $\overline{\Phi}$ and satisfies $(s\Phi_{\sigma})N_{\sigma}(s\Phi_{\sigma})^{-1} = qN_{\sigma}$, so the fiber of $M_{\overline{N}}$ over N_{σ} has an \mathcal{O} -point. The relative dimension of the formally smooth $\mathrm{Nil}_{\overline{N}}$ is $\dim G_k - \dim Z_{G_k}(\overline{N})$ by Lemma 6.4, and $M_{\overline{N}} \rightarrow \mathrm{Nil}_{\overline{N}}$ is a $\widehat{Z_G(N_{\sigma})}$ -torsor since it has an \mathcal{O} -point over N_{σ} . As $Z_G(N_{\sigma})$ is smooth it follows that $M_{\overline{N}}$ is formally smooth over $\mathrm{Spf} \mathcal{O}$ of relative dimension $\dim G_k$. \square

Example 6.14. This recovers [CHT08, Lemma 2.4.19] in the case $G = \mathrm{GL}_n$.

Let S be the (torus) quotient of G by its derived group G' , and $\mu : G \rightarrow S$ the quotient map. For use later, we now study a variant where we fix a lift $\nu : T_q \rightarrow S(\mathcal{O})$ of $\mu \circ \overline{\rho} : T_q \rightarrow S(k)$:

Corollary 6.15. *The deformation condition of minimally ramified lifts $\rho : T_q \rightarrow G(R)$ satisfying $\mu \circ \rho = \nu$ is a liftable deformation condition. The framed deformation ring $R_{\overline{\rho}}^{\mathrm{m.r.}, \nu, \square}$ is of relative dimension $\dim G_k - 1$.*

Proof. The quotient torus $S = G/G'$ is split of rank 1, so the subscheme $R_{\overline{\rho}}^{\mathrm{m.r.}, \nu, \square} \subset R_{\overline{\rho}}^{\mathrm{m.r.}, \square}$ is the vanishing of locus of a single function. As $R_{\overline{\rho}}^{\mathrm{m.r.}, \square}$ is formally smooth over \mathcal{O} with relative dimension $\dim G_k$, it suffices to check that the tangent space of $R_{\overline{\rho}}^{\mathrm{m.r.}, \nu, \square}$ over k (in the sense of [Maz97, §15]) is a proper subspace of the tangent space of $R_{\overline{\rho}}^{\mathrm{m.r.}, \square}$.

Let Z be the maximal central torus of G . On the level of Lie algebras, we know that $\mathrm{Lie} G$ splits over \mathcal{O} as a direct sum of $\mathrm{Lie} G'$ and $\mathrm{Lie} S \simeq \mathrm{Lie} Z$ as p is very good for G . We can modify a lift ρ_0 over $R = k[\epsilon]/(\epsilon^2)$ by multiplying against an unramified non-trivial character $T_q \rightarrow Z(R)$ with trivial reduction, changing $\mu \circ \rho_0$. Thus the tangent space of $R_{\overline{\rho}}^{\mathrm{m.r.}, \nu, \square}$ is a proper subspace of that of $R_{\overline{\rho}}^{\mathrm{m.r.}, \square}$. \square

6.4. Deformation Conditions Based on Parabolic Subgroups. The use of nilpotent orbits is not the only approach to defining a deformation condition at ramified places not above p . As discussed in §1.3, the method used to prove [CHT08, Lemma 2.4.19] suggests a generalization from GL_n to other groups G based on deformations lying in certain parabolic subgroups of G . This deformation condition is not smooth for algebraic groups beyond GL_n , so it does not work in Ramakrishna’s method. In this section we give a conceptual explanation for this phenomenon.

Let $P \subset G$ be a parabolic \mathcal{O} -subgroup. The Richardson orbit for P_k intersects $(\mathrm{Lie} P)_k$ in a dense open set which is a single geometric orbit under P_k . Suppose that $\overline{\rho}(\tau)$ is the exponential of a k -point \overline{N} in the Richardson orbit, and consider deformations $\rho : T_q \rightarrow G(\mathcal{O})$ of $\overline{\rho}$ ramified with respect to P in the sense that $\rho(\tau) \in P$ (compare with Definition 1.4). This requires specifying a lift of \overline{N} that lies in $\mathrm{Lie} P$. One could hope that such lifts would automatically be $G(\mathcal{O})$ -conjugate to the fixed lift N_{σ} defined in Proposition 4.12, reminiscent of the definition we gave for $\mathrm{Nil}_{\overline{N}}$, a situation in which the associated (framed) deformation ring is smooth.

We now show that often smoothness fails if \overline{N} does *not* lie in the Richardson orbit of P_k . Lifts of \overline{N} can “change nilpotent type” yet still lie in a parabolic lifting P_k , such as the example of the standard Borel subgroup in GL_3 with

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix} \quad \text{lifting} \quad \overline{N} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In particular, we easily obtain non-pure nilpotents. This is very bad: the nilpotent orbits over a field are smooth but the nilpotent cone is not smooth, so the deformation problem of deforming

with respect to P should not be smooth because “it sees multiple orbits”. Furthermore, even if we could lift $\overline{\rho}(\tau)$ appropriately, there would still be problems lifting $\overline{\rho}(\phi)$ because the centralizer of a non-pure nilpotent is not smooth over \mathcal{O} (the special and generic fiber typically have different dimensions). So it is crucial to choose a parabolic such that \overline{N} lies in the Richardson orbit of P_k .

For GL_n , all nilpotent orbits are Richardson orbits. This is not true in general. In particular, we should not expect the deformation condition of being ramified with respect to a parabolic to be liftable. Example 1.5 illustrates this phenomenon for GSp_4 , which we now revisit in a more conceptual manner.

Example 6.16. Take $G = \mathrm{GSp}_4$. Parabolic subgroups correspond to isotropic flags. Up to conjugacy, these subgroups are G (the trivial parabolic) and stabilizers of the flags

$$\begin{aligned} 0 \subset \mathrm{Span}(v_1) \subset \mathrm{Span}(v_1, v_2) \subset \mathrm{Span}(v_1, v_2, v_3) \subset k^4, \\ 0 \subset \mathrm{Span}(v_1) \subset \mathrm{Span}(v_1, v_2, v_3) \subset k^4, \quad 0 \subset \mathrm{Span}(v_1, v_2) \subset k^4 \end{aligned}$$

where $\{v_1, v_2, v_3, v_4\}$ is a basis of k^4 . Their Richardson orbits correspond to the nilpotent orbits indexed respectively by the partitions $1 + 1 + 1 + 1$, 4 , 4 , and $2 + 2$. In particular, the same Richardson orbit is associated to two of these. There is one more partition of 4 with odd numbers appearing an even number of times: $2 + 1 + 1$. This corresponds to a nilpotent orbit that is not a Richardson orbit; for the representation in Example 1.5, $\log(\overline{\rho}(\tau))$ is in this nilpotent orbit.

7. MINIMALLY RAMIFIED DEFORMATIONS OF SYMPLECTIC AND ORTHOGONAL GROUPS

We continue the notation of the previous section. We have defined the minimally ramified deformation condition for representations factoring through the quotient $T_q = \widehat{\mathbf{Z}} \rtimes \mathbf{Z}_p$ of the tame Galois group Γ_L^t at a place away from p . In this section, we will adapt the matrix-theoretic methods in [CHT08, §2.4.4], making use of more conceptual module-theoretic arguments, to define the minimally ramified deformation condition for any representation when $G = \mathrm{GSp}_m$ or $G = \mathrm{GO}_m$. (Minor variants of this method work for Sp_m and SO_m , and the original method of [CHT08, §2.4.4] works for GL_m .) We naturally embed G into $\mathrm{GL}(M)$ for a free \mathcal{O} -module M of rank m , and let V denote the reduction of M , a vector space over the residue field k .

We consider a representation $\overline{\rho}: \Gamma_L \rightarrow G(k) \subset \mathrm{GL}(V)(k)$ which may be wildly ramified (with L an ℓ -adic field for $\ell \neq p$). We will define a deformation condition for $\overline{\rho}$ in terms of the minimally ramified deformation condition for certain associated tamely ramified representations, after possibly extending \mathcal{O} . In §7.1, we analyze $\overline{\rho}$ as being built out of two pieces of data: representations of a closed normal subgroup Λ_L of Γ_L whose pro-order is prime to p , and tamely ramified representations of Γ_L/Λ_L . The representation theory of Λ_L is manageable since its pro-order is prime to p , and representations of Γ_L/Λ_L can be understood using the results of the previous section.

7.1. Decomposing Representations. We begin with a few preliminaries about representations over rings. Let Λ' be a profinite group and R be an Artinian coefficient ring with residue field k . If Λ' has pro-order prime to p , the representation theory over k is nice: every finite-dimensional continuous representation is a direct sum of irreducibles, and every such representation is projective over $k[\Lambda']$ for any finite discrete quotient Λ of Λ' through which the representation factors. We are also interested in corresponding statements over an Artinian coefficient ring R .

Fact 7.1. *Suppose the pro-order of Λ' is prime to p . Let P and P' be $R[\Lambda']$ -modules that are finitely generated over R with continuous action of Λ' , and F be a $k[\Lambda']$ -module that is finite dimensional over k with continuous action of Λ' . Let Λ be a finite discrete quotient of Λ' through which the Λ' -actions on P , P' , and F factor.*

- (1) *If P is free as an R -module, it is projective as a $R[\Lambda]$ -module.*
- (2) *If P and P' are projective over $R[\Lambda]$, they are isomorphic if and only if \overline{P} and $\overline{P'}$ are isomorphic.*

(3) *There exists a projective $R[\Lambda]$ -module (unique up to isomorphism) whose reduction is F .*

These statements are special cases of results in [Ser77, §14.4]. We now record two lemmas which do not need the assumption that the pro-order of Λ' is prime to p .

Lemma 7.2. *Let P and P' be $R[\Lambda']$ -modules, finitely generated over R with continuous action of Λ' factoring through a finite discrete quotient Λ of Λ' . Assume P and P' are $R[\Lambda]$ -projective. The natural map gives an isomorphism*

$$\mathrm{Hom}_{\Lambda'}(P, P') \otimes_R k \rightarrow \mathrm{Hom}_{\Lambda'}(\overline{P}, \overline{P'}).$$

Proof. We may replace $\mathrm{Hom}_{\Lambda'}$ with Hom_{Λ} . Note that $\mathfrak{m}P' = \mathfrak{m} \otimes_R P'$, so $\mathrm{Hom}_{\Lambda}(P, \mathfrak{m}P') = \mathrm{Hom}_{\Lambda}(P, P') \otimes_R \mathfrak{m}$ as P and P' are $R[\Lambda]$ -projective. Then apply $\mathrm{Hom}_{\Lambda}(P, -)$ to the exact sequence $0 \rightarrow \mathfrak{m}P' \rightarrow P' \rightarrow P'/\mathfrak{m}P' \rightarrow 0$. \square

Lemma 7.3. *Let Λ be a finite group and let M and M' be finite $R[\Lambda]$ -modules whose reductions \overline{M} and \overline{M}' are non-isomorphic irreducible $k[\Lambda]$ -modules. Then $\mathrm{Hom}_{R[\Lambda]}(M, M') = 0$.*

Proof. Filter M' by the composition series $\{\mathfrak{m}^i M'\}$, and consider the surjection

$$\mathfrak{m}^i/\mathfrak{m}^{i+1} \otimes \overline{M}' \twoheadrightarrow \mathfrak{m}^i M'/\mathfrak{m}^{i+1} M'.$$

The action of Λ on $\mathfrak{m}^i/\mathfrak{m}^{i+1} \otimes \overline{M}'$ is solely on the irreducible \overline{M}' , so as a $k[\Lambda]$ -module $\mathfrak{m}^i M'/\mathfrak{m}^{i+1} M'$ is isomorphic to a direct sum of copies of \overline{M}' . Thus

$$\mathrm{Hom}_{R[\Lambda]}(M, \mathfrak{m}^i M'/\mathfrak{m}^{i+1} M') = \mathrm{Hom}_{k[\Lambda]}(\overline{M}, \mathfrak{m}^i M'/\mathfrak{m}^{i+1} M') = 0$$

as \overline{M} and \overline{M}' are non-isomorphic $k[\Lambda]$ -modules.

By descending induction on i , we shall show that

$$\mathrm{Hom}_{R[\Lambda]}(M, \mathfrak{m}^i M') = 0.$$

For large i , $\mathfrak{m}^i M' = 0$. Consider the exact sequence

$$0 \rightarrow \mathfrak{m}^{i+1} M' \rightarrow \mathfrak{m}^i M' \rightarrow \mathfrak{m}^i M'/\mathfrak{m}^{i+1} M' \rightarrow 0.$$

Applying $\mathrm{Hom}_{R[\Lambda]}(M, -)$, we obtain a left exact sequence

$$0 \rightarrow \mathrm{Hom}_{R[\Lambda]}(M, \mathfrak{m}^{i+1} M') \rightarrow \mathrm{Hom}_{R[\Lambda]}(M, \mathfrak{m}^i M') \rightarrow \mathrm{Hom}_{R[\Lambda]}(M, \mathfrak{m}^i M'/\mathfrak{m}^{i+1} M')$$

The left term is 0 by induction, and the right term is 0 by the above calculation. This completes the induction. \square

Given $\overline{\rho} : \Gamma_L \rightarrow G(k) \subset \mathrm{GL}(V)(k)$ and a lift $\rho : \Gamma_L \rightarrow G(R) \subset \mathrm{GL}(M)(R)$ for some $R \in \mathcal{C}_{\mathcal{O}}$, we now turn to decomposing the $R[\Gamma_L]$ -module M . Let $I_L \subset \Gamma_L$ be the inertia group, and pick a surjection $I_L \rightarrow \mathbf{Z}_p$. Define Λ_L to be the kernel of this surjection (normal in Γ_L). This is a pro-finite group with pro-order prime to p , and is independent of the choice of surjection. Define the quotient

$$T_L := \Gamma_L / \Lambda_L,$$

which is a quotient of the tamely ramified Galois group Γ_L^t and of the form $T_q = \widehat{\mathbf{Z}} \rtimes \mathbf{Z}_p$ as in §6. We wish to compatibly decompose V and M as Λ_L -modules and then understand the action of Γ_L on the decomposition.

We first make a finite extension of k (and of \mathcal{O}) so that all of the (finitely many) irreducible representations of Λ_L over k occurring in V are absolutely irreducible over k .

Because Λ_L has order prime to p , $\mathrm{Res}_{\Lambda_L}^{\Gamma_L}(V)$ is completely reducible and we can write

$$\mathrm{Res}_{\Lambda_L}^{\Gamma_L}(V) = \bigoplus_{\tau} V_{\tau}$$

where τ runs through the set of isomorphism classes of irreducible representations of Λ_L over k occurring in V , and each V_τ is the τ -isotypic component. We will obtain an analogous decomposition for M .

Let Γ be a finite discrete quotient of Γ_L through which ρ factors, and let Λ be the image of Λ_L in Γ . Using Fact 7.1(3) we can lift τ to a projective $R[\Lambda]$ -module $\tilde{\tau}$ unique up to isomorphism. We will eventually want this lift to have additional properties (see §7.2), but this is not yet necessary. We set $W_\tau := \text{Hom}_{\Lambda_L}(\tilde{\tau}, M)$ and consider the natural morphism

$$\bigoplus_{\tau} \tilde{\tau} \otimes_R W_\tau \rightarrow M.$$

Note that M is $R[\Lambda]$ -projective by Fact 7.1(1).

Lemma 7.4. *This map is an isomorphism of $R[\Lambda_L]$ -modules.*

Proof. It suffices to check it is an isomorphism of $R[\Lambda]$ -modules. When $R = k$, $\text{End}_\Lambda(\tau) = k$ as we extended k so that all of the irreducible representations of Λ over k occurring inside V are absolutely irreducible. Splitting up V as a direct sum of irreducibles, we obtain the desired isomorphism.

In the general case, the map is an isomorphism after reducing modulo \mathfrak{m} (use Lemma 7.2). Thus by Nakayama's lemma it is surjective. Since M is R -projective, the formation of the kernel commutes with reduction modulo \mathfrak{m} . Thus, again using Nakayama's lemma the kernel is zero. \square

We define M_τ to be the image of $\tilde{\tau} \otimes_R \text{Hom}_{\Lambda_L}(\tilde{\tau}, M)$ in M . It is the largest $R[\Lambda_L]$ -direct summand whose reduction is a direct sum of copies of τ .

We next seek to understand the action of Γ_L on this canonical decomposition of M . For $g \in \Gamma_L$, consider the $R[\Lambda_L]$ -module gM_τ : it is a direct summand of M over R whose reduction is a direct sum of copies of the representation τ^g defined by $\tau^g(h) = \tau(g^{-1}hg)$ for $h \in \Lambda_L$. Thus we see that $gM_\tau = M_{\tau^g}$ inside M , and Γ_L permutes the M_τ 's. The orbits corresponds to sets of conjugate representations.

Consider the stabilizer of V_τ :

$$\Gamma_{L,\tau} = \{g \in \Gamma_L : gV_\tau = V_\tau \text{ inside } V\} = \{g \in \Gamma_L : \tau^g \simeq \tau\} \subset \Gamma_L$$

with corresponding image

$$\Gamma_\tau = \{g \in \Gamma : gV_\tau = V_\tau \text{ inside } V\} = \{g \in \Gamma : \tau^g \simeq \tau\} \subset \Gamma.$$

Then the k -span of the Γ_L -orbit of V_τ is exactly the representation $\text{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} V_\tau = \text{Ind}_{\Gamma_\tau}^\Gamma V_\tau$. Letting $[\tau]$ denote the set of $R[\Lambda]$ -isomorphism classes of Λ -representations Γ -conjugate to τ , by taking into account the action of Γ_τ the isomorphism in Lemma 7.4 becomes an isomorphism of $R[\Gamma_L]$ -modules

$$(7.1) \quad \bigoplus_{[\tau]} \text{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} M_\tau \xrightarrow{\sim} M$$

using one representative τ per Γ_L -conjugacy class $[\tau]$.

For orthogonal or symplectic representations, this decomposition interacts with duality. Denote the similitude character by μ , and let $\bar{\nu} := \mu \circ \bar{\rho} : \Gamma_L \rightarrow k^\times$. Let N be a free \mathcal{O} -module of rank 1 on which Γ_L acts by a specified continuous \mathcal{O}^\times -valued lift ν of $\bar{\nu}$, and let \bar{N} be its reduction modulo \mathfrak{m} . For an \mathcal{O} -module M , define $M^\vee = \text{Hom}_{\mathcal{O}}(M, N)$ with the evident Γ_L -action. The perfect pairing gives an isomorphism of $R[\Gamma_L]$ -modules $M \simeq M^\vee$. In particular,

$$M_\tau^\vee \simeq M_{\tau^*}$$

for some irreducible representation τ^* of Λ_L occurring in V . Note that $\tau^* \simeq \tau^\vee$ as $k[\Lambda_L]$ -modules. There are three cases:

- **Case 1:** τ is not conjugate to τ^* ;
- **Case 2:** τ is isomorphic to τ^* ;

• **Case 3:** τ is conjugate to τ^* but not isomorphic.

In the second case, we claim that the isomorphism of $k[\Lambda_L]$ -modules $\iota : \tau \simeq \tau^\vee$ gives a sign-symmetric perfect pairing on τ . Note that $W_\tau = \text{Hom}_\Lambda(\tau, V) \simeq \text{Hom}_\Lambda(\tau^\vee, V^\vee) \simeq W_\tau^\vee$ as $V_\tau \simeq V_\tau^\vee$. This isomorphism φ_τ defines a pairing $\langle, \rangle_{W_\tau}$ on W_τ via

$$\langle w_1, w_2 \rangle_{W_\tau} := \varphi_\tau(w_1)(w_2).$$

Let $\psi : V \rightarrow V^\vee$ be the isomorphism given by $m \mapsto \langle m, - \rangle_V$, and define $\langle v_1, v_2 \rangle_\tau := \iota(v_1)(v_2)$ for $v_1, v_2 \in \tau$. We have a commutative diagram

$$\begin{array}{ccccc} \tau \otimes W_\tau & \xrightarrow{\text{id} \otimes \varphi_\tau} & \tau \otimes W_\tau^\vee & \xrightarrow{\iota \otimes \text{id}} & \tau^\vee \otimes W_\tau^\vee \\ \downarrow & & & & \downarrow \\ V_\tau & \xrightarrow{\psi} & & & V_\tau^\vee \end{array}$$

The commutativity says that for elementary tensors $m_i = v_i \otimes w_i \in V_\tau = \tau \otimes W_\tau$ we have

$$\begin{aligned} \langle m_1, m_2 \rangle_M &= \psi(m_1)(m_2) = (\iota(v_1) \otimes \varphi_\tau(w_1))(v_2 \otimes w_2) \\ (7.2) \quad &= \iota(v_1)(v_2) \cdot \varphi_\tau(w_1)(w_2) = \langle v_1, v_2 \rangle_\tau \langle w_1, w_2 \rangle_{W_\tau}. \end{aligned}$$

Lemma 7.5. *The pairing \langle, \rangle_τ is a sign-symmetric.*

Proof. Suppose there exists $v \in \tau$ such that $\iota(v)(v) \neq 0$. For $w_1, w_2 \in W_\tau$, (7.2) gives

$$\iota(v)(v) \varphi_\tau(w_1)(w_2) = \langle v \otimes w_1, v \otimes w_2 \rangle_V = \epsilon \langle v \otimes w_2, v \otimes w_1 \rangle_V = \epsilon \iota(v)(v) \varphi_\tau(w_2)(w_1).$$

Canceling $\iota(v)(v)$, we conclude that $\langle w_1, w_2 \rangle_{W_\tau} = \epsilon \langle w_2, w_1 \rangle_{W_\tau}$. Using (7.2), we conclude that

$$\epsilon \iota(v_2)(v_1) \cdot \varphi_\tau(w_2)(w_1) = \epsilon \langle m_2, m_1 \rangle_V = \langle m_1, m_2 \rangle_V = \iota(v_1)(v_2) \cdot \varphi_\tau(w_1)(w_2) = \epsilon \iota(v_1)(v_2) \cdot \varphi_\tau(w_2)(w_1).$$

Choosing w_1 and w_2 with $\langle w_2, w_1 \rangle_{W_\tau} \neq 0$ (possible as \langle, \rangle_V is perfect), we then conclude that $\langle v_1, v_2 \rangle_\tau = \langle v_2, v_1 \rangle_\tau$.

Otherwise $\iota(v)(v) = 0$ for all $v \in \tau$, in which case \langle, \rangle_τ is alternating. \square

In §7.2 we will see that the action of Λ_L on the module underlying $\tilde{\tau}$ can be extended to an action of $\Gamma_{L,\tau}$ factoring through Γ_τ . Therefore, $W_\tau = \text{Hom}_{\Lambda_L}(\tilde{\tau}, M)$ is naturally a representation of $T_{L,\tau} := \Gamma_{L,\tau}/\Lambda_L$, and of $T_\tau := \Gamma_\tau/\Lambda$ (a finite quotient of $T_{L,\tau}$). In §7.4, we will use the minimally ramified deformation condition of §6 to specify which deformations W_τ are allowed. Together with the decomposition (7.1)

$$\bigoplus_{[\tau]} \text{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} (\tilde{\tau} \otimes W_\tau) \rightarrow M$$

this defines a deformation condition for $\bar{\rho}$. Some care is needed to ensure compatibility with the pairing on M , which will require breaking into cases in the next sections.

7.2. Extension of Representations. We continue the notation of the previous section, where τ is an absolutely irreducible representation of Λ_L over k . We need to lift this to a representation over \mathcal{O} and extend it to a representation of $\Gamma_{L,\tau}$. We will have to do something extra for the representation to be compatible with a pairing, depending on how τ and τ^* are related.

In **Case 1**, we ignore the pairing. The results of [CHT08, §2.4.4] let us pick a $\mathcal{O}[\Gamma_{L,\tau}]$ -module $\tilde{\tau}$ that is a free \mathcal{O} -module and reduces to τ . In this case, $\tilde{\tau}^\vee$ is a free \mathcal{O} -module reducing to τ^* .

In **Case 2**, from Lemma 7.5 it follows that τ is a symplectic or orthogonal representation. We will adapt the GL_n -technique of [CHT08] to produce a symplectic or orthogonal extension $\tilde{\tau}$. Letting $n = \dim \tau$, the representation τ gives a homomorphism $\tau : \Lambda_L \rightarrow G(k)$ where G is GSp_n or GO_n .

First, we claim that there is a continuous lift $\tilde{\tau} : \Lambda_L \rightarrow G(W(k))$: without the pairing, this would be Fact 7.1(3). To also take into account the pairing, consider deformation theory for the residual representation τ . This is a smooth deformation condition as $H^2(\Lambda_L, \text{ad } \tau) = 0$: Λ_L has pro-order

prime to p and $\text{ad } \tau$ has order a power of p . Therefore the desired lift exists. It is unique (up to conjugations which preserve τ) because the tangent space is zero dimensional as $H^1(\Lambda_L, \text{ad } \tau) = 0$. By considering representations of the group $\Lambda_L / \ker(\tau)$, we may and do assume that $\ker(\tilde{\tau}) = \ker(\tau)$ as subgroups of Λ_L .

Remark 7.6. For $g \in \Gamma_{L,\tau}$, the isomorphism of $k[\Lambda_L]$ -modules $\tau^g \simeq \tau$ lifts to an isomorphism $\tilde{\tau}^g \simeq \tilde{\tau}$ of $\mathcal{O}[\Lambda_L]$ -modules by uniqueness. Thus $\Gamma_{L,\tau} = \{g \in \Gamma_L : \tilde{\tau}^g \simeq \tilde{\tau}\}$.

We will now show how to continuously extend $\tilde{\tau}$ to $\Gamma_{L,\tau}$. The first step in constructing the extension is to understand the structure of $\Gamma_{L,\tau}$ and $I_L \cap \Gamma_{L,\tau}$, where I_L is the inertia group.

Recall that $T_L = \Gamma_L / \Lambda_L$ is the semi-direct product of $\hat{\mathbf{Z}}$ and \mathbf{Z}_p , where $\hat{\mathbf{Z}}$ is generated by a lift of Frobenius ϕ and \mathbf{Z}_p is generated by an element σ , with $\phi\sigma\phi^{-1} = \sigma^q$ where $q = \ell^a$ is the size of the residue field of L .

Lemma 7.7. *The exact sequence*

$$1 \rightarrow \Lambda_L \rightarrow \Gamma_L \rightarrow T_L \rightarrow 1$$

is topologically split, so Γ_L is a semi-direct product.

Proof. Let S be a Sylow pro- p subgroup of I_L , which must be isomorphic to \mathbf{Z}_p . Let ϕ be a lift of Frobenius to Γ_L . Then $\phi S \phi^{-1}$ is another Sylow pro- p subgroup of I_L , and hence is conjugate to S by an element of I_L . By choosing the lift ϕ , we may thereby assume that ϕ normalizes S . But then it is clear that S and ϕ together topologically generate T_L , giving the desired splitting. \square

For $T_{L,\tau} := \Gamma_{L,\tau} / \Gamma_L$, this gives a topological splitting of

$$1 \rightarrow \Lambda_L \rightarrow \Gamma_{L,\tau} \rightarrow T_{L,\tau} \rightarrow 1.$$

As $\Gamma_{L,\tau}$ is an open subgroup of Γ_L , we observe that $T_{L,\tau}$ is an open subgroup of T_L . Note that $T_{L,\tau}$ is normal and topologically generated by some powers of ϕ and σ which will be denoted by ϕ_τ and σ_τ (since any open subgroup of a semidirect product $C \ltimes C'$ for pro-cyclic C and C' is of the form $C_0 \ltimes C'_0$ for open subgroups $C_0 \subset C$ and $C'_0 \subset C'$). In particular, $T_{L,\tau}$ is itself isomorphic to $T_{q'}$ for some q' . The element σ_τ and Λ_L together topologically generate $\Gamma_{L,\tau} \cap I_L$.

Before extending $\tilde{\tau}$, we need several technical lemmas.

Lemma 7.8. *The centralizer of the image of $\tilde{\tau}$ is \mathcal{O} .*

Proof. As τ is absolutely irreducible, $\text{End}_{\Lambda_L}(\tau) = k$. By Lemma 7.2, we see that the reduction of $\text{End}_{\Lambda_L}(\tilde{\tau})$ modulo the maximal ideal of \mathcal{O} is k , so the map $\mathcal{O} \hookrightarrow \text{End}_{\Lambda_L}(\tilde{\tau})$ is surjective by Nakayama's lemma. \square

Lemma 7.9. *The dimension of τ is not divisible by p .*

Proof. As τ is continuous and Λ_L has pro-order prime to p , the representation τ factors through a finite discrete quotient Λ of Λ_L whose order is prime to p . Such a representation is the reduction of a projective $\mathcal{O}[\Lambda]$ -module by Fact 7.1(3). Inverting p , we obtain a representation of Λ in characteristic zero that is absolutely irreducible since the “reduction” τ is absolutely irreducible over k . By [Ser77, §6.5 Corollary 2], the dimension of this representation (equal to the dimension of τ) divides the order of Λ . \square

We will now extend $\tilde{\tau}$ from $\Lambda_L \subset I_L$ to $\Gamma_{L,\tau}$ by defining it on the topological generators σ_τ and ϕ_τ . We say that such an extension has *tame determinant* if $\det(\tilde{\tau}(\sigma_\tau))$ has finite order which is prime to p .

Lemma 7.10. *There is a unique continuous extension $\tilde{\tau} : \Gamma_{L,\tau} \cap I_L \rightarrow G(\mathcal{O})$ with tame determinant.*

Proof. A continuous extension of $\tilde{\tau}$ to $\Gamma_{L,\tau} \cap I_L$ is determined by its value on σ_τ . As $\sigma_\tau \in \Gamma_{L,\tau}$, in light of Remark 7.6 there is an $A \in G(\mathcal{O})$ such that for $g \in \Lambda_L$ we have

$$\tilde{\tau}(\sigma_\tau g \sigma_\tau^{-1}) = A \tilde{\tau}(g) A^{-1}.$$

We would like to send σ_τ to the element A . However, this might not produce a continuous extension, and even if it does it might not have tame determinant unless we pick A correctly. As σ_τ is a topological generator for a group isomorphic to \mathbf{Z}_p , the continuity of the extension with $\sigma_\tau \mapsto A$ is equivalent to some p -power of A having trivial reduction. We wish to show that there is a unique choice of such A that also makes the extension have tame determinant.

We will first show that some power A^{p^b} lies in the centralizer of the image $\tilde{\tau}(\Lambda_L)$. Consider the conjugation action of $\langle \sigma_\tau \rangle$ on Λ_L . As $\ker \tilde{\tau} = \ker \tau$ is a normal subgroup of $\Gamma_{L,\tau}$ (if $g \in \Gamma_{L,\tau}$ and $\tau(g) = 1$, then $\tau^g(h)$ is conjugate to $\tau(h) = 1$ by Remark 7.6) we get an action of $\langle \sigma_\tau \rangle$ on $\Lambda_L / \ker \tau \simeq \tau(\Lambda_L)$. The action is continuous, so there is a power p^b such that for all $g \in \Lambda_L$ we have

$$\tau(\sigma_\tau^{p^b} g \sigma_\tau^{-p^b}) = \tau(g).$$

As $\ker \tilde{\tau} = \ker \tau$, we see that

$$A^{p^b} \tilde{\tau}(g) A^{-p^b} = \tilde{\tau}(\sigma_\tau^{p^b} g \sigma_\tau^{-p^b}) = \tilde{\tau}(g).$$

Therefore A^{p^b} lies in the centralizer of $\tilde{\tau}(\Lambda_L)$.

By Lemma 7.8, this centralizer is \mathcal{O} . We claim that by multiplying A by some unit in \mathcal{O} , we can arrange for the continuous extension $\tilde{\tau}$ to exist and have tame determinant. We will use the fact that an element of \mathcal{O}^\times is the product of a 1-unit and a Teichmüller lift of an element of k^\times . As $A^{p^b} \in \mathcal{O}^\times$ and the p th power map is an automorphism of k^\times , we may multiply A by a unit scalar so that A^{p^b} reduces to the identity matrix. By Lemma 7.9, the dimension n of τ is prime to p so we may multiply A by a 1-unit so that $\det(A)$ has finite order prime to p . Sending σ_τ to this particular A gives a continuous extension with tame determinant.

Let's show this extension is unique. Any extension must send σ_τ to an element of the form wA for $w \in \mathcal{O}^\times$ (the centralizer of the image $\tilde{\tau}(\Lambda_L)$). By continuity, there is a power p^b such that $(wA)^{p^b}$ reduces to the identity. This means that w^{p^b} reduces to the identity, and hence that w reduces to the identity. On the other hand, $\det(wA) \det(A)^{-1} = w^n$. The left side has finite order that is relatively prime to p , so w^n does too. This forces $w^n = 1$ since its reduction is 1. But as n is prime to p (Lemma 7.9), the only n th roots of unity in \mathcal{O}^\times are Teichmüller lifts. Therefore $w = 1$. \square

Lemma 7.11. *There is a continuous extension $\tilde{\tau} : \Gamma_{L,\tau} \rightarrow G(\mathcal{O})$.*

Proof. We extend $\tilde{\tau}$ in Lemma 7.10 continuously to $\Gamma_{L,\tau}$ by defining it on ϕ_τ . As $\phi_\tau \in \Gamma_{L,\tau}$, there is an element $A \in G(\mathcal{O})$ conjugating $\tilde{\tau} : \Lambda_L \rightarrow G(\mathcal{O})$ to $\tilde{\tau}^{\phi_\tau} : \Lambda_L \rightarrow G(\mathcal{O})$. Each has a unique extension to a continuous morphism from $I_L \cap \Gamma_{L,\tau}$ to $G(\mathcal{O})$ with tame determinant. Therefore for $g \in I_L \cap \Gamma_{L,\tau}$ we have

$$\tilde{\tau}(\phi_\tau g \phi_\tau^{-1}) = A \tilde{\tau}(g) A^{-1}$$

since the right side has the same (tame) determinant as $\tilde{\tau}$ on $I_L \cap T_\tau$. We can continuously extend $\tilde{\tau} : I_L \cap \Gamma_{L,\tau} \rightarrow G(\mathcal{O})$ by sending ϕ_τ to A since A has reduction with finite order. \square

This gives the desired lift and extension of τ in the case that $\tau \simeq \tau^*$.

In **Case 3**, τ is conjugate to τ^* but not isomorphic. The argument follows the same structure as the previous case, but we make a few modifications to treat $\tau \oplus \tau^*$ together. In particular, we can pick a copy of the $k[\Lambda_L]$ -module τ inside V and a copy of $\tau^* \simeq \tau^\vee$ inside V such that the pairing restricted to $\tau \oplus \tau^*$ is perfect.

Define $\Gamma_{L,\tau\oplus\tau^*} = \{g \in \Gamma_L : (\tau \oplus \tau^*)^g \simeq \tau \oplus \tau^*\}$. It contains $\Gamma_{L,\tau}$ with index 2, as conjugation either preserves τ and τ^* or swaps them. Arguing as in the paragraph after Lemma 7.7, we obtain a split exact sequence

$$0 \rightarrow \Lambda_L \rightarrow \Gamma_{L,\tau\oplus\tau^*} \rightarrow T_{L,\tau\oplus\tau^*} \rightarrow 1$$

where $T_{L,\tau\oplus\tau^*}$ is an open normal subgroup of T_L topologically generated by some powers of ϕ and σ which we denote by $\phi_{\tau\oplus\tau^*}$ and $\sigma_{\tau\oplus\tau^*}$. We may arrange that either

- **Case 3a:** $\phi_{\tau\oplus\tau^*}^2 = \phi_\tau$ and $\sigma_{\tau\oplus\tau^*} = \sigma_\tau$ or
- **Case 3b:** $\phi_{\tau\oplus\tau^*} = \phi_\tau$ and $\sigma_{\tau\oplus\tau^*}^2 = \sigma_\tau$.

In **Case 3a**, we begin by lifting τ to \mathcal{O} as a representation of Λ_L : as before, we do this using the fact that the pro-order of Λ_L is prime to p , and obtain a lift $\tilde{\tau}$ unique up to isomorphism. We extend $\tilde{\tau}$ to be a representation of $\Gamma_{L,\tau} \cap I_L$ by defining it on σ_τ using the GL_n -version of Lemma 7.10, [CHT08, Lemma 2.4.11]. There it is shown all such extensions are unique up to equivalence. In particular, $\tilde{\tau}$ and $(\tilde{\tau}^{\phi_{\tau\oplus\tau^*}})^\vee$ are isomorphic $\mathcal{O}[\Gamma_{L,\tau} \cap I_K]$ -modules. We can use this to define a sign-symmetric perfect pairing on $\tilde{\tau} \oplus \tilde{\tau}^{\phi_{\tau\oplus\tau^*}}$ that is compatible with the action of $\Gamma_{L,\tau} \cap I_K$ and $\phi_{\tau\oplus\tau^*}$, hence of $\Gamma_{L,\tau\oplus\tau^*}$.

In **Case 3b**, as τ^\vee and $\tau^{\sigma_{\tau\oplus\tau^*}}$ are isomorphic $k[\Lambda_L]$ -modules it follows that $\tilde{\tau}^\vee$ and $\tilde{\tau}^{\sigma_{\tau\oplus\tau^*}}$ are isomorphic $\mathcal{O}[\Lambda_L]$ -modules. In particular, this isomorphism gives a natural way to define a sign-symmetric perfect pairing on $M = \tilde{\tau} \oplus \tilde{\tau}^{\sigma_{\tau\oplus\tau^*}}$ lifting the residual one. This pairing is compatible with the action of $\Gamma_{L,\tau\oplus\tau^*} \cap I_L$ (which is generated by Λ_L and $\sigma_{\tau\oplus\tau^*}$). Finally, we claim that M and M^{ϕ_τ} are isomorphic. As $\phi_\tau \in \Gamma_{L,\tau}$ preserves τ , the reductions of M and M^{ϕ_τ} are isomorphic by an isomorphism which identifies τ and τ^{ϕ_τ} . By uniqueness of the lift of τ as a $\mathcal{O}[\Lambda_L]$ -module, we obtain an isomorphism of $\tilde{\tau}$ and $\tilde{\tau}^{\phi_\tau}$ and hence of M and M^{ϕ_τ} compatible with the pairing. Then we proceed as in the proof of Lemma 7.11, defining an image of ϕ_τ using this isomorphism.

In conclusion, we have shown:

Lemma 7.12. *In case 3, there exists an $\mathcal{O}[\Gamma_{L,\tau\oplus\tau^*}]$ -module $\widetilde{\tau \oplus \tau^*}$ with pairing lifting $\tau \oplus \tau^*$ together with its pairing.*

7.3. Lifts with Pairings. We continue the notation of §7.1, and analyze how the duality pairing interacts with the decomposition (7.1). Recall that we obtained an isomorphism $M \simeq M^\vee$ of $R[\Gamma_L]$ -modules which gave isomorphisms $M_\tau \simeq M_\tau^\vee$ of $R[\Gamma_{L,\tau}]$ -modules. The key point is that for any lift and extension τ' of τ to an $\mathcal{O}[\Gamma_{L,\tau}]$ -module, the isomorphism of $R[\Lambda_L]$ -modules

$$\tau' \otimes \mathrm{Hom}_{\Lambda_L}(\tau', M) \rightarrow M_\tau$$

is compatible with the $\Gamma_{L,\tau}$ -action.

To do this, it is convenient to break into the cases introduced at the end of §7.1. For an irreducible $k[\Lambda]$ -module τ occurring in V , note that $(\tau^g)^\vee = (\tau^\vee)^g$ for any $g \in \Gamma_L$, so if $\tau \simeq \tau^*$ then $\tau^g \simeq (\tau^g)^*$. We let

- Σ_n denote the set of Γ_L -conjugacy classes of such τ for which τ is not conjugate to τ^* ;
- Σ_e denote the set of Γ_L -conjugacy classes of such τ for which $\tau \simeq \tau^*$;
- Σ_c denote the set of Γ_L -conjugacy classes of such τ for which τ^* is conjugate to τ but $\tau \not\simeq \tau^*$.

From (7.1), we obtain a decomposition

$$(7.3) \quad M = \bigoplus_{\tau \in \Sigma_n} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} (\tau' \otimes W_\tau) \oplus \bigoplus_{\tau \in \Sigma_e} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} (\tau' \otimes W_\tau) \oplus \bigoplus_{\tau \in \Sigma_c} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} (\tau' \otimes W_\tau)$$

where τ' is any lift and extension of τ to Γ_τ and $W_\tau = \mathrm{Hom}_\Lambda(\tau', M)$ is a representation of $T_{L,\tau}$. Note that W_τ is free as an R -module (since M and τ' are, with $\tau' \neq 0$ and R local), and hence that W_τ is tamely ramified of the type considered in §6.

We may rewrite this to make use of the special extensions constructed in §7.2. In particular, for $\tau \in \Sigma_c$ we rewrite

$$\mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L}(\tau' \otimes W_\tau) = \mathrm{Ind}_{\Gamma_{L,\tau \oplus \tau^*}}^{\Gamma_L}(\widetilde{\tau \oplus \tau^*} \otimes W_{\tau \oplus \tau^*})$$

where $W_{\tau \oplus \tau^*} := \mathrm{Hom}_{\Lambda_L}(\widetilde{\tau \oplus \tau^*}, M)$. This uses the notation and results from Case 3 in §7.2, in particular the fact that $\tau \oplus \tau^*$ is an irreducible representation of the group generated by Λ_L and a $g \in \Gamma_L$ with $\tau^* \simeq \tau^g$. Note that $W_{\tau \oplus \tau^*}$ is a representation of $T_{L,\tau \oplus \tau^*}$, which is a subgroup of $T_L = \Gamma_L / \Lambda_L$, hence of the form T_q as considered in §6. Using the extensions $\tilde{\tau}$ and $\widetilde{\tau \oplus \tau^*}$ from Cases 1 and 2 from §7.2, we obtain a decomposition

$$(7.4) \quad M = \bigoplus_{\tau \in \Sigma_n} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L}(\tilde{\tau} \otimes W_\tau) \oplus \bigoplus_{\tau \in \Sigma_e} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L}(\tilde{\tau} \otimes W_\tau) \oplus \bigoplus_{\tau \in \Sigma_c} \mathrm{Ind}_{\Gamma_{L,\tau \oplus \tau^*}}^{\Gamma_L}(\widetilde{\tau \oplus \tau^*} \otimes W_{\tau \oplus \tau^*}).$$

Now let M' be another $R[\Gamma_L]$ -module that is finite free over R such that the irreducible representations of Λ_L occurring in $V' := M' / \mathfrak{m}M'$ are among the same τ 's, so

$$M' = \bigoplus_{\tau \in \Sigma_n} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L}(\tilde{\tau} \otimes W'_\tau) \oplus \bigoplus_{\tau \in \Sigma_e} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L}(\tilde{\tau} \otimes W'_\tau) \oplus \bigoplus_{\tau \in \Sigma_c} \mathrm{Ind}_{\Gamma_{L,\tau \oplus \tau^*}}^{\Gamma_L}(\widetilde{\tau \oplus \tau^*} \otimes W'_{\tau \oplus \tau^*}).$$

Lemma 7.13. *The natural map*

$$\bigoplus_{\tau \in \Sigma_n} \mathrm{Hom}_{T_{L,\tau}}(W_\tau, W'_\tau) \oplus \bigoplus_{\tau \in \Sigma_e} \mathrm{Hom}_{T_{L,\tau}}(W_\tau, W'_\tau) \oplus \bigoplus_{\tau \in \Sigma_c} \mathrm{Hom}_{T_{L,\tau \oplus \tau^*}}(W_{\tau \oplus \tau^*}, W'_{\tau \oplus \tau^*}) \rightarrow \mathrm{Hom}_{\Gamma_L}(M, M')$$

is an isomorphism.

Proof. We may immediately pass to working with representations of the finite discrete groups Γ and Λ . Notice that

$$\mathrm{Hom}_\Gamma(\mathrm{Ind}_{\Gamma_\tau}^\Gamma(M_\tau), \mathrm{Ind}_{\Gamma_\tau}^\Gamma(M'_\tau)) \simeq \mathrm{Hom}_{\Gamma_\tau}(M_\tau, \mathrm{Ind}_{\Gamma_\tau}^\Gamma(M'_\tau)) \simeq \mathrm{Hom}_{\Gamma_\tau}(M_\tau, M'_\tau)$$

where the second isomorphism uses that $\mathrm{Hom}_{\Gamma_\tau}(M_\tau, M'_{\tau^g}) = 0$ by Lemma 7.3 when τ and τ^g are non-isomorphic. Furthermore, if τ_1 and τ_2 are not Γ -conjugate then

$$\mathrm{Hom}_\Gamma(\mathrm{Ind}_{\Gamma_{\tau_1}}^\Gamma(M_{\tau_1}), \mathrm{Ind}_{\Gamma_{\tau_2}}^\Gamma(M'_{\tau_2})) = 0$$

again using Lemma 7.3. Then using (7.1) we see that

$$\mathrm{Hom}_\Gamma(M, M') = \bigoplus_{[\tau_1], [\tau_2]} \mathrm{Hom}_\Gamma(\mathrm{Ind}_{\Gamma_{\tau_1}}^\Gamma(M_{\tau_1}), \mathrm{Ind}_{\Gamma_{\tau_2}}^\Gamma(M'_{\tau_2})) = \bigoplus_{[\tau]} \mathrm{Hom}_{\Gamma_\tau}(M_\tau, M'_\tau).$$

All the irreducible finite-dimensional representations of Λ occurring in V and V' are absolutely irreducible over k by design. For $\tau \in \Sigma_n \cup \Sigma_e$, consider the natural inclusion

$$(7.5) \quad \mathrm{Hom}_R(W_\tau, W'_\tau) \hookrightarrow \mathrm{Hom}_\Lambda(\tilde{\tau} \otimes W_\tau, \tilde{\tau} \otimes W'_\tau) = \mathrm{Hom}_\Lambda(\tilde{\tau}, \tilde{\tau}) \otimes_R \mathrm{Hom}_R(W_\tau, W'_\tau),$$

using that W_τ and W'_τ are R -free of finite rank and Λ has no effect on them. But $R \hookrightarrow \mathrm{Hom}_\Lambda(\tilde{\tau}, \tilde{\tau})$ is an isomorphism because $\mathrm{End}_\Lambda(\tau) = k$ and because surjectivity can be checked modulo \mathfrak{m}_R using Lemma 7.2. As $M_\tau \simeq \tilde{\tau} \otimes W_\tau$, this implies that

$$\begin{aligned} \mathrm{Hom}_{\Gamma_\tau}(M_\tau, M'_\tau) &= \mathrm{Hom}_\Lambda(M_\tau, M'_\tau)^{T_\tau} = \mathrm{Hom}_\Lambda(\tilde{\tau} \otimes W_\tau, \tilde{\tau} \otimes W'_\tau)^{T_\tau} \\ &= \mathrm{Hom}_R(W_\tau, W'_\tau)^{T_\tau} = \mathrm{Hom}_{T_\tau}(W_\tau, W'_\tau) \end{aligned}$$

where T_τ is the image of $T_{L,\tau}$ in Γ_τ . An analogous computation in the case $\tau \in \Sigma_c$ completes the proof. \square

We can now consider the duality isomorphism $M \simeq M^\vee$. By Lemma 7.13, this is equivalent to a collection of isomorphisms of $R[T_{L,\tau}]$ -modules $\varphi_\tau : W_\tau \simeq W_{\tau^*}^\vee$ for $\tau \in \Sigma_e \cup \Sigma_n$ and an isomorphism of $R[T_{L,\tau \oplus \tau^*}]$ -modules $\varphi_\tau : W_{\tau \oplus \tau^*} \simeq W_{\tau \oplus \tau^*}^\vee$ for $\tau \in \Sigma_c$. We analyze the cases separately.

In **Case 1** ($\tau \in \Sigma_n$), note that $\text{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} M_\tau$ is an isotropic subspace of M as $\tau \not\simeq \tau^*$. In particular, the perfect sign-symmetric pairing on $\text{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} M_\tau \oplus \text{Ind}_{\Gamma_{L,\tau^*}}^{\Gamma_L} M_{\tau^*}$ is equivalent to an isomorphism of $R[\Gamma_L]$ -modules

$$\text{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} M_\tau \simeq \left(\text{Ind}_{\Gamma_{L,\tau^*}}^{\Gamma_L} M_{\tau^*} \right)^\vee,$$

which is equivalent to the isomorphism of $R[T_{L,\tau}]$ -modules $\varphi_\tau : W_\tau \simeq W_{\tau^*}^\vee$. (Note that the similitude character ν is present in the use of the dual.)

In **Case 2** ($\tau \in \Sigma_e$), the perfect sign-symmetric pairing on $\text{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} M_\tau$ is equivalent to an isomorphism $W_\tau \simeq W_\tau^\vee$ of $R[T_{L,\tau}]$ -modules. Thus it gives a pairing $\langle \cdot, \cdot \rangle_{W_\tau}$ on W_τ via

$$\langle w_1, w_2 \rangle_{W_\tau} := \varphi_\tau(w_1)(w_2).$$

We claim this pairing is sign-symmetric.

From §7.2 we have an isomorphism $\iota : \tilde{\tau} \simeq \tilde{\tau}^\vee$ of $R[\Gamma_{L,\tau}]$ -modules. As at the end of §7.1, let $\psi : M \rightarrow M^\vee$ be the isomorphism of $R[\Gamma_L]$ -modules given by $m \mapsto \langle m, - \rangle_M$, and define $\langle v_1, v_2 \rangle_{\tilde{\tau}} := \iota(v_1)(v_2)$. We have a commutative diagram

$$\begin{array}{ccccc} \tilde{\tau} \otimes W_\tau & \xrightarrow{\text{id} \otimes \varphi_\tau} & \tilde{\tau} \otimes W_\tau^\vee & \xrightarrow{\iota \otimes \text{id}} & \tilde{\tau}^\vee \otimes W_\tau^\vee \\ \downarrow & & & & \downarrow \\ M_\tau & \xrightarrow{\psi} & & & M_\tau^\vee \end{array}$$

The commutativity says that for an elementary tensor $m_i = v_i \otimes w_i \in M_\tau = \tilde{\tau} \otimes W_\tau$ we have

$$\begin{aligned} \langle m_1, m_2 \rangle_M &= \psi(m_1)(m_2) = (\iota(v_1) \otimes \varphi_\tau(w_1))(v_2 \otimes w_2) \\ (7.6) \quad &= \iota(v_1)(v_2) \cdot \varphi_\tau(w_1)(w_2) = \langle v_1, v_2 \rangle_{\tilde{\tau}} \langle w_1, w_2 \rangle_{W_\tau}. \end{aligned}$$

The pairings are perfect and $\langle \cdot, \cdot \rangle_{\tilde{\tau}}$ is sign-symmetric, so the pairing on W_τ is sign-symmetric if and only if the pairing on M_τ is sign-symmetric.

In **Case 3** ($\tau \in \Sigma_c$), an analogous argument using the isomorphism $\widetilde{\tau \oplus \tau^*} \simeq \widetilde{\tau \oplus \tau^*}^\vee$ of $R[\Gamma_{L,\tau \oplus \tau^*}]$ -modules shows that the pairing induced by $\varphi_\tau : W_{\tau \oplus \tau^*} \simeq W_{\tau \oplus \tau^*}^\vee$ is sign-symmetric if and only if the pairing on

$$\text{Ind}_{\Gamma_{L,\tau \oplus \tau^*}}^{\Gamma_L} \left(\widetilde{\tau \oplus \tau^*} \otimes W'_{\tau \oplus \tau^*} \right)$$

induced from the pairing on M is sign-symmetric.

7.4. Minimally Ramified Deformations. We can now define the minimally ramified deformation condition for $\bar{\rho} : \Gamma_L \rightarrow G(k)$, under the continuing assumption that we have extended k so all irreducible representations of Λ_L occurring in V are absolutely irreducible over k . From (7.4), we obtain a decomposition

$$(7.7) \quad V = \bigoplus_{\tau \in \Sigma_n} \text{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} (\tilde{\tau} \otimes \overline{W}_\tau) \oplus \bigoplus_{\tau \in \Sigma_e} \text{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} (\tilde{\tau} \otimes \overline{W}_\tau) \oplus \bigoplus_{\tau \in \Sigma_c} \text{Ind}_{\Gamma_{L,\tau \oplus \tau^*}}^{\Gamma_L} \left(\widetilde{\tau \oplus \tau^*} \otimes \overline{W}_{\tau \oplus \tau^*} \right),$$

where \overline{W}_τ is a representation of $T_{L,\tau}$ over k and $\overline{W}_{\tau \oplus \tau^*}$ is a representation of $T_{L,\tau \oplus \tau^*}$.

If $\tau \in \Sigma_n$, define $\overline{G}_\tau := \underline{\text{Aut}}(\overline{W}_\tau)$. If $\tau \in \Sigma_e$, there is a sign-symmetric perfect pairing $\langle \cdot, \cdot \rangle_{\overline{W}_\tau}$ on \overline{W}_τ : in that case define $\overline{G}_\tau := \underline{\text{GAut}}(\overline{W}_\tau, \langle \cdot, \cdot \rangle_{\overline{W}_\tau})$. (The notation $\underline{\text{GAut}}$ means automorphisms preserving the pairing up to scalar.) If $\tau \in \Sigma_c$, there is a sign-symmetric perfect pairing on $\overline{W}_{\tau \oplus \tau^*}$: in that case define $\overline{G}_\tau := \underline{\text{GAut}}(\overline{W}_{\tau \oplus \tau^*}, \langle \cdot, \cdot \rangle_{\overline{W}_{\tau \oplus \tau^*}})$. Make a finite extension of k so that all the

pairings are split. Lift \overline{G}_τ to a split reductive group G_τ over \mathcal{O} by lifting the split linear algebra data.

Definition 7.14. Let $\rho : \Gamma_L \rightarrow G(R)$ be a continuous Galois representation lifting $\overline{\rho}$ as above, with associated $R[\Gamma]$ -module

$$M = \bigoplus_{\tau \in \Sigma_n} \text{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} (\tilde{\tau} \otimes W_\tau) \oplus \bigoplus_{\tau \in \Sigma_e} \text{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} (\tilde{\tau} \otimes W_\tau) \oplus \bigoplus_{\tau \in \Sigma_c} \text{Ind}_{\Gamma_{L,\tau \oplus \tau^*}}^{\Gamma_L} \left(\widetilde{\tau \oplus \tau^*} \otimes W_{\tau \oplus \tau^*} \right).$$

We say that ρ is *minimally ramified* with similitude character ν if each W_τ and $W_{\tau \oplus \tau^*}$ is minimally ramified in the sense of Definition 6.11 as a representation of $T_{L,\tau}$ or $T_{L,\tau \oplus \tau^*}$ valued in the group G_τ with specified similitude character. (Note that defining the minimally ramified deformation condition as in §6 may require an additional étale local extension of \mathcal{O} , which as always is harmless for applications.)

Let $D_{\overline{\rho}}^{\text{m.r.}, \nu}$ denote the deformation functor for $\overline{\rho}$ with specified similitude character ν , and $\mathcal{D}_{G_\tau}^{\text{m.r.}, \nu}$ (respectively $D_{G_\tau}^{\text{m.r.}, \nu}$) denote the deformation functor for \overline{W}_τ or $\overline{W}_{\tau \oplus \tau^*}$ viewed as a representation valued in G_τ (respectively with specified similitude character ν). In particular, letting $r = \dim \overline{W}_\tau$ (or $\dim \overline{W}_{\tau \oplus \tau^*}$ when $\tau \in \Sigma_c$), we have that the adjoint representation $\text{ad } \overline{W}_\tau$ is the Lie algebra of \overline{G}_τ , which is the Lie algebra of GSp_r or GO_r when $\tau \in \Sigma_e$ or Σ_c , and the Lie algebra of GL_r when $\tau \in \Sigma_n$. Let Σ'_n consist of one representative for each pair of representations $\tau, \tau^* \in \Sigma_n$.

Proposition 7.15. *The natural map*

$$D_{\overline{\rho}}^{\text{m.r.}, \nu}(R) \rightarrow \prod_{\tau \in \Sigma'_n} D_{G_\tau}^{\text{m.r.}, \nu}(R) \times \prod_{\tau \in \Sigma_e} D_{G_\tau}^{\text{m.r.}, \nu}(R) \times \prod_{\tau \in \Sigma_c} D_{G_\tau}^{\text{m.r.}, \nu}(R)$$

is an isomorphism.

Proof. This expresses the decomposition obtained in this section: given a lift ρ of $\overline{\rho}$, we obtain a decomposition of M as in Definition 7.14. Our analysis with pairings shows that when $\tau \in \Sigma_e$, W_τ is a deformation of \overline{W}_τ together with its sign-symmetric perfect pairing. Likewise, when $\tau \in \Sigma_c$ we know that $W_{\tau \oplus \tau^*}$ is a deformation of $\overline{W}_{\tau \oplus \tau^*}$ together with its pairing. When $\tau \in \Sigma_n$, we know $W_\tau \simeq W_{\tau^*}^\vee$. This gives the natural map: to $\rho \in D_{\overline{\rho}}^{\text{m.r.}, \nu}(R)$ associate the collection of the W_τ for $\tau \in \Sigma_e \cup \Sigma_c \cup \Sigma'_n$.

Conversely, given W_τ for $\tau \in \Sigma_e \cup \Sigma_c \cup \Sigma'_n$, and defining $W_{\tau^*} := W_\tau^\vee$ for $\tau \in \Sigma'_n$ we can define a lift

$$M := \bigoplus_{\tau \in \Sigma_n} \text{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} (\tilde{\tau} \otimes W_\tau) \oplus \bigoplus_{\tau \in \Sigma_e} \text{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} (\tilde{\tau} \otimes W_\tau) \oplus \bigoplus_{\tau \in \Sigma_c} \text{Ind}_{\Gamma_{L,\tau \oplus \tau^*}}^{\Gamma_L} \left(\widetilde{\tau \oplus \tau^*} \otimes W_{\tau \oplus \tau^*} \right).$$

as in (7.1). (Note that the groups $\Gamma_{L,\tau}$ depend only on the fixed residual representation V .) For $\tau \in \Sigma_e$, the sign-symmetric perfect pairing on the lift W_τ gives an isomorphism $\varphi_\tau : W_\tau \simeq W_\tau^\vee$ of $R[T_{L,\tau}]$ -modules, which gives a sign-symmetric pairing on $\text{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} (\tilde{\tau} \otimes W_\tau)$ (using (7.6)). Likewise, $\tau \in \Sigma_c$ the sign-symmetric pairing on $W_{\tau \oplus \tau^*}$ gives one on $\text{Ind}_{\Gamma_{L,\tau \oplus \tau^*}}^{\Gamma_L} \left(\widetilde{\tau \oplus \tau^*} \otimes W_{\tau \oplus \tau^*} \right)$. For $\tau \in \Sigma_n$, we obtain an isomorphism $\varphi_\tau : W_\tau \simeq W_{\tau^*}^\vee$ of $R[T_{L,\tau}]$ -modules and hence a sign-symmetric perfect pairing on $(\tilde{\tau} \otimes W_\tau) \oplus (\tilde{\tau}^\vee \otimes W_{\tau^*})$ which gives one on $\text{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} (\tilde{\tau} \otimes W_\tau) \oplus \text{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} (\tilde{\tau}^\vee \otimes W_{\tau^*})$. Putting these together, we obtain a sign-symmetric pairing on M ; the action of Γ_L preserves it up to scalar, giving a continuous homomorphism $\rho : \Gamma_L \rightarrow G(R)$.

Finally, we claim that these constructions are compatible with strict equivalence of lifts, giving an identification of the deformation functors. For $g \in \widehat{G}(R)$, decompose the g -conjugate Γ_L -representation M^g according to (7.4). As g reduces to the identity, it must respect the decomposition into τ -isotypic pieces, so gives automorphisms $g_\tau \in \text{Aut}(W_\tau)$ and $g_\tau \in \text{Aut}(W_{\tau \oplus \tau^*})$. If $\tau \in \Sigma_e$ or Σ_c , as g is compatible with the pairing on M we see g_τ is compatible with the pairing

as well. For $\tau \in \Sigma_e$, the g_τ -conjugate $T_{L,\tau}$ -representation $W_\tau^{g_\tau}$ is minimally ramified as minimally ramified lifts of \overline{W}_τ for the group $T_{L,\tau}$ are a deformation condition, and likewise for $\tau \in \Sigma_c$ and $\tau \in \Sigma'_n$.

Conversely, given $g_\tau \in \text{Aut}(W_\tau)$ reducing to the identity (compatible with the pairing on W_τ or $W_{\tau \oplus \tau^*}$ if there is one), using (7.1) and acting on each piece we obtain a lift of the form M^g for $g \in \widehat{G}(R)$. Thus the identification is compatible with strict equivalence. \square

Corollary 7.16. *The minimally ramified deformation condition with fixed similitude character is liftable. The dimension of the tangent space is $h^0(\Gamma_L, \text{ad}^0(\overline{\rho}))$.*

Proof. Liftability is a consequence of Proposition 7.15 and the smoothness of the framed minimally ramified deformation ring for representations of $T_{L,\tau}$ (Proposition 6.13 and Corollary 6.15). Recall that the dimension of the tangent space of a deformation condition of a representation $\theta : T_{L,\tau} \rightarrow G_\tau(k)$ is the dimension of the tangent space of the framed deformation ring minus the relative dimension of G_τ plus the dimension of $H^0(T_{L,\tau}, \text{ad } \theta)$ (see Remark 2.4). By Corollary 6.15, for $\tau \in \Sigma_e$ the dimension of the tangent space of $D_{G_\tau}^{\text{m.r.}, \nu}$ is $h^0(T_{L,\tau}, \text{ad } \overline{W}_\tau) - 1 = h^0(T_{L,\tau}, \text{ad}^0 \overline{W}_\tau)$, and for $\tau \in \Sigma_c$ the dimension is $h^0(T_{L,\tau \oplus \tau^*}, \text{ad } \overline{W}_{\tau \oplus \tau^*}) - 1 = h^0(T_{L,\tau \oplus \tau^*}, \text{ad}^0 \overline{W}_{\tau \oplus \tau^*})$. For $\tau \in \Sigma'_n$, by Proposition 6.13 the dimension of the tangent space of $D_{G_\tau}^{\text{m.r.}}$ is $h^0(T_{L,\tau}, \text{ad } \overline{W}_\tau)$. Using Proposition 7.15, we see that the dimension of the tangent space of the minimally ramified deformation condition is

$$\sum_{\tau \in \Sigma_e} h^0(T_{L,\tau}, \text{ad}^0 \overline{W}_\tau) + \sum_{\tau \in \Sigma_c} h^0(T_{L,\tau \oplus \tau^*}, \text{ad}^0 \overline{W}_{\tau \oplus \tau^*}) + \sum_{\tau \in \Sigma'_n} h^0(T_{L,\tau}, \text{ad } \overline{W}_\tau).$$

It remains to identify this quantity with $h^0(\Gamma_L, \text{ad}^0(\overline{\rho}))$. Using Lemma 7.13

$$\begin{aligned} H^0(\Gamma_L, \text{End}(V)) &= \text{Hom}_{k[\Gamma_L]}(V, V) \\ &= \bigoplus_{\tau \in \Sigma_e \cup \Sigma_n} \text{Hom}_{T_{L,\tau}}(\overline{W}_\tau, \overline{W}_\tau) \oplus \bigoplus_{\tau \in \Sigma_c} \text{Hom}_{T_{L,\tau \oplus \tau^*}}(\overline{W}_{\tau \oplus \tau^*}, \overline{W}_{\tau \oplus \tau^*}) \\ &= \bigoplus_{\tau \in \Sigma_e \cup \Sigma_n} H^0(T_{L,\tau}, \text{End}(\overline{W}_\tau)) \oplus \bigoplus_{\tau \in \Sigma_c} H^0(T_{L,\tau \oplus \tau^*}, \text{End}(\overline{W}_{\tau \oplus \tau^*})). \end{aligned}$$

We are interested in $H^0(\Gamma_L, \text{ad}^0(\overline{\rho}))$: the elements $\psi \in H^0(\Gamma_L, \text{End}(V))$ compatible with the pairing on V in the sense that for $v, v' \in V$

$$\langle \psi v, \psi v' \rangle = \langle v, v' \rangle.$$

The pairing on $V_\tau = \tau \otimes \overline{W}_\tau$ is induced by the pairings on \overline{W}_τ and τ when $\tau \in \Sigma_e$, and is induced by the pairings on $\overline{W}_{\tau \oplus \tau^*}$ and $\tau \oplus \tau^*$ when $\tau \in \Sigma_c$. When $\tau \in \Sigma'_n$, the pairing on $V_\tau \oplus V_{\tau^*}$ comes from the $\Gamma_{L,\tau}$ -isomorphism $V_\tau \simeq V_{\tau^*}^\vee$ which in turn comes from the $T_{L,\tau}$ -isomorphism $\overline{W}_\tau \simeq \overline{W}_{\tau^*}^\vee$. So ψ is compatible with the pairing if and only if

- when $\tau \in \Sigma_e$, the associated $\psi_\tau \in H^0(T_{L,\tau}, \text{End}(\overline{W}_\tau))$ is compatible with the pairing on \overline{W}_τ ;
- when $\tau \in \Sigma_c$, the associated $\psi_\tau \in H^0(T_{L,\tau \oplus \tau^*}, \text{End}(\overline{W}_{\tau \oplus \tau^*}))$ is compatible with the pairing on $\overline{W}_{\tau \oplus \tau^*}$;
- when $\tau \in \Sigma'_n$, the associated ψ_τ and ψ_{τ^*} are identified by duality and the isomorphism $\overline{W}_\tau \simeq \overline{W}_{\tau^*}^\vee$.

In the first two cases, $\text{ad}^0 \overline{W}_\tau$ and $\text{ad}^0 \overline{W}_{\tau \oplus \tau^*}$ are the symplectic or orthogonal Lie algebra, consisting exactly of endomorphisms compatible with the pairing on \overline{W}_τ . In the third, we just choose one of ψ_τ and ψ_{τ^*} without restriction, which determines the other. Thus we see

$$H^0(\Gamma_L, \text{ad}^0(\overline{\rho})) = \bigoplus_{\tau \in \Sigma_e} H^0(T_{L,\tau}, \text{ad}^0 \overline{W}_\tau) \oplus \bigoplus_{\tau \in \Sigma_c} H^0(T_{L,\tau \oplus \tau^*}, \text{ad}^0 \overline{W}_{\tau \oplus \tau^*}) \oplus \bigoplus_{\tau \in \Sigma'_n} H^0(T_{L,\tau}, \text{ad } \overline{W}_\tau). \quad \square$$

8. FONTAINE-LAFFAILLE THEORY WITH PAIRINGS

We begin by establishing some notation and reviewing the key results of Fontaine-Laffaille theory. It was first studied by Fontaine and Laffaille [FL82], who introduced a contravariant functor relating torsion-crystalline representations and Fontaine-Laffaille modules. For deformation theory, in particular compatibility with tensor products, it is necessary to use a covariant version, introduced in [BK90]. The details of relating this covariant functor to the functor studied by Fontaine and Laffaille that are omitted in [BK90] are written down in [Con94]. We then study Fontaine-Laffaille modules with the extra data of a pairing by analyzing tensor products and duals, in preparation for studying the Fontaine-Laffaille deformation condition in §9. This analysis generalizes unpublished results in [Pat06].

8.1. Covariant Fontaine-Laffaille Theory. Let $K = W(k')[\frac{1}{p}]$ for a perfect field k' of characteristic p . Let $W = W(k')$ and $\sigma : W \rightarrow W$ denote the Frobenius map. Recall that a *torsion-crystalline* representation with Hodge-Tate weights in $[a, b]$ is a $\mathbf{Z}_p[\Gamma_K]$ -module T for which there exists a crystalline representation V with Hodge-Tate weights in $[a, b]$ and Γ_K -stable lattices $\Lambda \subset \Lambda'$ in V such that Λ'/Λ is isomorphic to T . Our convention will be that the Hodge-Tate weight of the cyclotomic character is -1 , which will work well with covariant functors. The analogue of torsion-crystalline representations on the semilinear algebra side are certain classes of Fontaine-Laffaille modules:

Definition 8.1. A *Fontaine-Laffaille* module is a W -module M together with a decreasing filtration $\{M^i\}_{i \in \mathbf{Z}}$ of M by W -submodules and a family of σ -semilinear maps $\{\varphi_M^i : M^i \rightarrow M\}$ such that:

- The filtration is separated and exhaustive: $M = \cup_{i \in \mathbf{Z}} M^i$ and $\cap_{i \in \mathbf{Z}} M^i = 0$.
- For $m \in M^{i+1}$, $p \cdot \varphi_M^{i+1}(m) = \varphi_M^i(m)$.

Morphisms of Fontaine-Laffaille modules $f : M \rightarrow N$ are W -linear maps such that $f(M^i) \subset N^i$ and $f \circ \varphi_M^i = \varphi_N^i \circ f$ for all i . The category of Fontaine-Laffaille modules is denoted MF_W .

Let $\mathrm{MF}_{W, \mathrm{tor}}^f$ denote the full subcategory consisting of M for which M is of finite length (as a W -module) and for which $\sum_{i \in \mathbf{Z}} \varphi^i(M^i) = M$, and $\mathrm{MF}_{W, \mathrm{tor}}^{f, [a, b]}$ to be the full subcategory with the additional condition that $M^a = M$ and $M^{b+1} = 0$.

Remark 8.2. Jumps in the filtration will turn out to correspond Hodge-Tate weights, so the condition $M^a = M$ and $M^{b+1} = 0$ with $a \leq b$ corresponds to Hodge-Tate weights lying in $[a, b]$. We call the set of jumps in the filtration the *Fontaine-Laffaille weights*.

We are also interested in a variant that allows non-torsion modules.

Definition 8.3. A *filtered Dieudonné* module M is a Fontaine-Laffaille module for which the M^i are direct summands of M as W -modules and for which

$$\sum_{i \in \mathbf{Z}} \varphi^i(M^i) = M.$$

Let \mathcal{D}_K denote the full subcategory of MF_W consisting of filtered Dieudonné modules M for which $M^a = M$ and $M^{b+1} = 0$ for some $0 \leq b - a \leq p - 2$.

Note that for $M \in \mathrm{MF}_{W, \mathrm{tor}}^{f, [a, b]}$, it is automatic that M^i is a direct summand of M . There are natural notions of tensor products and duality.

Definition 8.4. For Fontaine-Laffaille modules M_1 and M_2 , define $M_1 \otimes M_2$ to have underlying W -module $M_1 \otimes M_2$, filtration given by $(M_1 \otimes M_2)^n = \sum_{i+j=n} M_1^i \otimes M_2^j$, and maps $\varphi_{M_1 \otimes M_2}^n$ induced by the $\varphi_{M_1}^i$ and $\varphi_{M_2}^j$.

Definition 8.5. For $M \in \mathrm{MF}_{W, \mathrm{tor}}^f$, define M^* to be $\mathrm{Hom}_W(M, K/W)$ with the dual filtration

$$M_i^* := \mathrm{Hom}_W(M/M^{1-i}, K/W)$$

and with $\varphi_{M^*}^i$ characterized for $f \in M_i^*$ and $m \in M^j$ by $\varphi_{M^*}^i(f)(\varphi^j(m)) = 0$ when $j \geq 1 - i$ and by $\varphi_{M^*}^i(f)(\varphi^j(m)) = f(p^{-i-j}m)$ when $j < 1 - i$ (in which case $-i - j \geq 0$).

Lemma 8.6. *There is a unique $(\varphi_{M^*}^i)$ satisfying these constraints. Using it, M^* is an object of $\mathrm{MF}_{W,\mathrm{tor}}^f$. Then $M \mapsto M^*$ is a contravariant functor from $\mathrm{MF}_{W,\mathrm{tor}}^f$ to itself, and $M \simeq M^{**}$ naturally in M .*

Proof. Uniqueness is immediate, while existence is checked in [Con94, §7.5]. We will use a similar argument in Lemma 8.21 and Lemma 8.22. \square

To connect Fontaine-Laffaille modules and torsion-crystalline representations, we use the period ring A_{cris} . For our purposes, what is important is that A_{cris} is a period ring that has an action of Γ_K , a ring endomorphism φ (coming from the p th power map) and a filtration $\{\mathrm{Fil}^i A_{\mathrm{cris}}\}$. In particular, it carries both an action of Γ_K and the structure of a Fontaine-Laffaille module. A convenient reference is [Hat, §2.2], which reviews A_{cris} for the purposes of constructing the contravariant Fontaine-Laffaille functor. We use it to define an analogue of V_{cris} :

Definition 8.7. For $M \in \mathrm{MF}_{W,\mathrm{tor}}^{f,[2-p,1]}$, define

$$T_{\mathrm{cris}}(M) := \ker(1 - \varphi_{A_{\mathrm{cris}} \otimes M}^0 : \mathrm{Fil}^0(A_{\mathrm{cris}} \otimes M) \rightarrow A_{\mathrm{cris}} \otimes M).$$

Remark 8.8. A small argument (see [Hat, §2.2]) also shows that

$$A_{\mathrm{cris},\infty} := A_{\mathrm{cris}} \otimes_W K/W = \varinjlim_n A_{\mathrm{cris}}/p^n A_{\mathrm{cris}} \in \mathrm{MF}_{W,\mathrm{tor}}^{f,[0,p-1]}.$$

This allows us define a contravariant functor from $\mathrm{MF}_{W,\mathrm{tor}}^{f,[0,p-1]}$ to $\mathrm{Rep}_{\mathbf{Z}_p}(\Gamma_K)$ by

$$T_{\mathrm{cris}}^*(M) := \mathrm{Hom}_{\mathrm{MF}_W}(M, A_{\mathrm{cris},\infty}).$$

This functor agrees with the functor U_S considered by Fontaine and Laffaille [Hat, Remark 2.7].

If $M \in \mathrm{MF}_{W,\mathrm{tor}}^{f,[2-p,1]}$ is killed by p , then

$$\begin{aligned} T_{\mathrm{cris}}^*(M^*) &= \mathrm{Hom}_{\mathrm{MF}_W}(M^*, A_{\mathrm{cris}}/pA_{\mathrm{cris}}) \\ &\simeq \ker(1 - \varphi_{A_{\mathrm{cris}} \otimes M}^0 : \mathrm{Fil}^0(A_{\mathrm{cris}} \otimes M) \rightarrow A_{\mathrm{cris}} \otimes M) \\ &= T_{\mathrm{cris}}(M) \end{aligned}$$

which is how Fontaine and Laffaille's results about T_{cris}^* imply results about T_{cris} .

We can extend T_{cris} to \mathcal{D}_K by defining an analogue of Tate-twisting:

Definition 8.9. For $M \in \mathrm{MF}_{W,\mathrm{tor}}^{f,[a,b]}$ and an integer s , define $M(s)$ to have the same underlying W -module with filtration $M(s)^i = M^{i-s}$ and maps $\varphi_{M(s)}^i = \varphi_M^{i-s}$.

Tate-twisting allows us to shift the weights and extend results in the range $[2 - p, 1]$ to any interval $[a, b]$ where $b - a \leq p - 2$. For $M \in \mathrm{MF}_{W,\mathrm{tor}}^{f,[a,b]}$, we define

$$T_{\mathrm{cris}}(M) = T_{\mathrm{cris}}(M(-(b-1)))(b-1)$$

Fact 8.10. *We have:*

- (1) *The covariant functor $T_{\mathrm{cris}} : \mathcal{D}_K \rightarrow \mathrm{Rep}_{\mathbf{Z}_p}[\Gamma_K]$ is well-defined, and is exact and fully faithfully.*
- (2) *For $M \in \mathcal{D}_K$, $T_{\mathrm{cris}}(M) = \varprojlim_n T_{\mathrm{cris}}(M/p^n M)$.*
- (3) *The essential image of $T_{\mathrm{cris}} : \mathrm{MF}_{W,\mathrm{tor}}^{f,[a,b]} \rightarrow \mathrm{Rep}_{\mathbf{Z}_p}[\Gamma_K]$ is stable under formation of sub-objects and quotients.*
- (4) *For $M \in \mathrm{MF}_{W,\mathrm{tor}}^{f,[a,b]}$, the length of M as a W -module is equal to the length of $T_{\mathrm{cris}}(M)$ as a \mathbf{Z}_p -module.*

- (5) For $M \in \mathcal{D}_K$, the Γ_K -representation $T_{\text{cris}}(M)[\frac{1}{p}]$ is crystalline.
- (6) Any torsion-crystalline $\mathbf{F}_p[\Gamma_K]$ -module \bar{V} whose Hodge-Tate weights lie in an interval of length $p-2$ is in the essential image of T_{cris} .

This is a modified version of [BK90, Theorem 4.3]. Additional details of the proof of that theorem are recorded in [Con94, §7]. The first, fourth, and fifth statements are stated explicitly in [BK90, Theorem 4.3]. The second is proven in [Con94, §7.2]. The claim about the essential image follows from the results of [Con94, §8.3-9.6]: a formal argument shows that if T_{cris}^* takes simple objects to simple objects, the essential image is stable under formation of sub-objects and quotients. The content is that T_{cris}^* takes simple objects to simple objects. The formal argument adapts to T_{cris} , and Remark 8.8 allows us to deduce that T_{cris} takes simple objects to simple objects as all simple objects are automatically killed by p . The last statement follows from relating T_{cris} to T_{cris}^* on p -torsion objects and the fact that for $r \in \{0, 1, \dots, p-2\}$, the functor T_{cris}^* induces an anti-equivalence between $\text{MF}_{W, \text{tor}}^{f, [0, r]}$ and the full subcategory of $\text{Rep}_{\mathbf{Z}_p}(\Gamma_K)$ consisting of torsion-crystalline Γ_K representations with Hodge-Tate weights in $[-r, 0]$.

Remark 8.11. Our convention that the Hodge-Tate weight of the cyclotomic character is -1 makes the Fontaine-Laffaille weights and Hodge-Tate weights match under T_{cris} .

8.2. Tensor Products and Freeness. We now address two properties of T_{cris} where it is crucial to be using the covariant functor. Definition 8.4 defined a tensor product for Fontaine-Laffaille modules. If $M_1 \in \text{MF}_{W, \text{tor}}^{f, [a_1, b_1]}$ and $M_2 \in \text{MF}_{W, \text{tor}}^{f, [a_2, b_2]}$, it is straightforward to verify that $M_1 \otimes M_2$ is an object of $\text{MF}_{W, \text{tor}}^{f, [a_1+a_2, b_1+b_2]}$. The functor T_{cris} is compatible with tensor products in the following sense:

Fact 8.12. Suppose that M_1 , M_2 , and $M_1 \otimes M_2$ each has Fontaine-Laffaille weights in an interval of length at most $p-2$. Then $T_{\text{cris}}(M_1) \otimes_{\mathbf{Z}_p} T_{\text{cris}}(M_2) \simeq T_{\text{cris}}(M_1 \otimes M_2)$.

There is a natural map from the left to the right coming from the multiplication of A_{cris} . To check this map is an isomorphism, one first checks on simple M_1 and M_2 using Fontaine and Laffaille's classification of simple Fontaine-Laffaille modules when the residue field k' is algebraically closed. This is explained in [Con94, §10.6]. Then one uses a dévissage argument to reduce to the general case, as explained in [Con94, §7.11].

Remark 8.13. An analogue of this compatibility is stated in [FL82, Remarques 6.13(b)] for the contravariant functor T_{cris}^* , but is missing a p -torsion hypothesis. In that case, we have

$$T_{\text{cris}}^*(M_1) = \text{Hom}_{\text{MF}_W}(M_1, A_{\text{cris}, \infty}) = \text{Hom}_{\text{MF}_W}(M_1, A_{\text{cris}}/pA_{\text{cris}})$$

and likewise for M_2 . Then multiplication on $A_{\text{cris}}/pA_{\text{cris}}$ gives a natural map

$$T_{\text{cris}}^*(M_1) \otimes T_{\text{cris}}^*(M_2) \rightarrow T_{\text{cris}}^*(M_1 \otimes M_2)$$

which can be checked to be an isomorphism by dévissage. But $A_{\text{cris}, \infty}$ is not a ring, so there is no natural map without a p -torsion hypothesis on M_1 and M_2 . This explains why it is crucial to work with the covariant functor T_{cris} .

For $M \in \text{MF}_{W, \text{tor}}^{f, [a, b]}$, if $V = T_{\text{cris}}(M)$ has “extra structure” then so does M . For example, if V were a deformation of a residual representation over a finite field k , V would be an $\mathcal{O} = W(k)$ -module. As T_{cris} is covariant and fully faithful, it is immediate that M is naturally an \mathcal{O} -module. The actions of \mathbf{Z}_p on M via the embeddings into \mathcal{O} and $W = W(k')$ are obviously compatible.

Recall that representations of Γ_K defined over a finite extension L of \mathbf{Q}_p can be viewed as \mathbf{Q}_p -vector spaces with the additional action of L . Assume there exists an embedding of K into L over \mathbf{Q}_p , so L splits K over \mathbf{Q}_p . These representations are modules over $L \otimes_{\mathbf{Q}_p} K \simeq \prod_{\tau: K \hookrightarrow L} L_{\tau}$ via $a \otimes b \mapsto (a\tau(b))$. For each \mathbf{Q}_p -embedding τ , there is a collection of Hodge-Tate weights.

We will generalize this structure to our setting: assume k' is finite, and more specifically that k' embeds in k , so $\mathcal{O}[\frac{1}{p}]$ splits the finite unramified K over \mathbf{Q}_p . Hence

$$\mathcal{O} \otimes_{\mathbf{Z}_p} W \simeq \prod_{\tau: W \hookrightarrow \mathcal{O}} \mathcal{O}_\tau$$

as \mathcal{O} -algebras, where τ varies over \mathbf{Z}_p -embeddings of W into \mathcal{O} and W acts on \mathcal{O}_τ via τ . We likewise obtain a decomposition of the $\mathcal{O} \otimes_{\mathbf{Z}_p} W$ -module M as

$$M = \bigoplus_{\tau: W \hookrightarrow \mathcal{O}} M_\tau.$$

Note that

$$\mathrm{Hom}_{\mathcal{O} \otimes_{\mathbf{Z}_p} W}(M, M') = \bigoplus_{\tau: W \hookrightarrow \mathcal{O}} \mathrm{Hom}_{\mathcal{O}}(M_\tau, M'_\tau).$$

Lemma 8.14. *If V is equipped with a Γ_K -equivariant \mathcal{O} -module structure then for $M_\tau^i := M_\tau \cap M^i$ we have*

$$M^i = \bigoplus_{\tau: W \hookrightarrow \mathcal{O}} M_\tau^i$$

and furthermore the σ -semilinear map $\varphi_M^i|_{M_\tau^i} : M_\tau^i \rightarrow M$ factors through $M_{\sigma\tau}$.

Proof. It is immediate that $M^i = \bigoplus_\tau M_\tau^i$, as the induced action of \mathcal{O} on M preserves the filtration. Let $j_i : M_\tau^i \rightarrow M^i$ be the inclusion, so $j_i(wm) = \tau(w)j_i(m)$ for $w \in W$ and $m \in M_\tau^i$. The fact that φ_M^i is σ -semilinear implies that for $m \in M_\tau^i$ and $w \in W$,

$$\sigma\tau(w)\varphi_M^i(j_i(m)) = \varphi_M^i(\tau(w)j_i(m)) = \varphi_M^i(j_i(wm)). \quad \square$$

Lemma 8.15. *The length of M as an \mathcal{O} -module equals the length of V as an \mathcal{O} -module multiplied by $[K : \mathbf{Q}_p]$.*

Proof. The length of M as a W -module equals the length of V as a \mathbf{Z}_p -module by Fact 8.10(4). The rest is bookkeeping. The \mathbf{F}_p -dimension of $\mathcal{O}/p\mathcal{O}$ is $t = [\mathcal{O}[\frac{1}{p}] : \mathbf{Q}_p]$, and the k' -dimension of $\mathcal{O}/p\mathcal{O}$ is $s = [\mathcal{O}[\frac{1}{p}] : K]$. It follows that

$$s \lg_{\mathcal{O}}(M) = \lg_W(M) = \lg_{\mathbf{Z}_p}(V) = t \lg_{\mathcal{O}}(V).$$

Hence $\lg_{\mathcal{O}}(M) = [K : \mathbf{Q}_p] \lg_{\mathcal{O}}(V)$. \square

We can prove a result about freeness when V and M are R -modules for an artinian coefficient \mathcal{O} -algebra R with residue field k :

Lemma 8.16. *We have V is a free R -module if and only if M is a free R -module. When M is a free R -module, all of the M_τ^i are free R -direct summands. All of the M_τ have the same rank.*

Proof. Let N be a finitely generated R -module with $n = \dim_k N/\mathfrak{m}_R N$. Then N is free if and only if $\lg_{\mathcal{O}}(N) = n \lg_{\mathcal{O}}(R)$, as we see via Nakayama's lemma applied to a map $R^n \rightarrow N$ inducing an isomorphism modulo \mathfrak{m}_R . From the exact sequence of Fontaine-Laffaille modules

$$0 \rightarrow \mathfrak{m}_R M \rightarrow M \rightarrow M/\mathfrak{m}_R M \rightarrow 0$$

and the fact that T_{cris} is covariant and exact, we see that $T_{\mathrm{cris}}(M/\mathfrak{m}_R M) = V/\mathfrak{m}_R V$. Using Lemma 8.15, if $\dim_k V/\mathfrak{m}_R V = n$ then $M/\mathfrak{m}_R M$ is a k -vector space of dimension $[K : \mathbf{Q}_p]n$. Thus to relate R -freeness of M and V we just need to show that $\lg_{\mathcal{O}}(M) = [K : \mathbf{Q}_p] \lg_{\mathcal{O}}(V)$, which is Lemma 8.15.

Now suppose M is a free R -module. By functoriality, the \mathbf{Z}_p -module direct summands M_τ of M are each R -submodules, so each M_τ is an R -module direct summand of M . Hence each M_τ is R -free when M is free. To deduce the same for each M_τ^i , we just need that each M_τ^i is an R -module

summand. By R -freeness of M , it suffices to show that each $M_\tau^i/\mathfrak{m}_R M_\tau^i \rightarrow M/\mathfrak{m}_R M$ is injective. Since M_τ^i is the “ τ -component” of M^i by Lemma 8.14 it is an R -module summand of M^i . Thus it suffices to show that

$$M^i/\mathfrak{m}_R M^i \rightarrow M/\mathfrak{m}_R M$$

is injective for all i . But this follows from the fact that $\mathrm{MF}_{W,\mathrm{tor}}^{f,[a,b]}$ is abelian.

To check that all of the M_τ have the same rank, by freeness it suffices to check that $\dim_k \overline{M}_\tau$ is independent of τ . As all \mathbf{Z}_p -embeddings of the *unramified* W into \mathcal{O} are of the form $\sigma^i \tau$ for some fixed \mathbf{Z}_p -embedding τ and σ has finite order, it suffices to show that

$$\dim_k \overline{M}_\tau \geq \dim_k \overline{M}_{\sigma\tau}.$$

As each \overline{M}_τ^i is a k -module direct summand of \overline{M}_τ , \overline{M}_τ is isomorphic to $\mathrm{gr}^\bullet \overline{M}_\tau$. But $\varphi_{\overline{M}}^i(\overline{M}^{i+1}) = 0$, so we obtain a map

$$\sum_i \varphi_{M_\tau}^i : \mathrm{gr}^\bullet \overline{M}_\tau \rightarrow \overline{M}_{\sigma\tau}.$$

As Fontaine-Laffaille modules satisfy

$$\overline{M} = \sum_i \varphi^i(\overline{M}^i)$$

the map $\sum_i \varphi_{M_\tau}^i$ is surjective. This completes the proof. \square

Remark 8.17. We get a set of Fontaine-Laffaille weights for each \mathbf{Z}_p -embedding $\tau : W \hookrightarrow \mathcal{O}$. We can also define the multiplicity of a weight w_τ to be the rank of the R -module $M_\tau^{w_\tau}/M_\tau^{w_\tau+1}$. The number of Fontaine-Laffaille weights (counted with multiplicity) is the same for each embedding. We say the Fontaine-Laffaille weights with respect to an embedding are distinct if each has multiplicity 1. This is analogous to the way a Hodge-Tate representation of Γ_K over a p -adic field splitting K over \mathbf{Q}_p has a set of Hodge-Tate weights for each \mathbf{Q}_p -embedding of K into that field.

We can now define a notion of a tensor product for Fontaine-Laffaille modules that are also R -modules for a coefficient ring R over \mathcal{O} .

Definition 8.18. Define $M_1 \otimes_R M_2$ to be the module $M_1 \otimes_R M_2$ together with filtration defined by $(M_1 \otimes_R M_2)^n = \sum_{i+j=n} M_1^i \otimes_R M_2^j$ and with $\varphi_{M_1 \otimes_R M_2}^n$ defined in the obvious way on the pieces.

Lemma 8.19. Suppose that M_1 , M_2 , and $M_1 \otimes_R M_2$ are all in $\mathrm{MF}_{W,\mathrm{tor}}^{f,[a,b]}$ with $0 \leq b - a \leq p - 2$. The natural map $T_{\mathrm{cris}}(M_1) \otimes_R T_{\mathrm{cris}}(M_2) \rightarrow T_{\mathrm{cris}}(M_1 \otimes_R M_2)$ is an isomorphism.

Proof. We have an exact sequence

$$0 \rightarrow J \rightarrow M_1 \otimes M_2 \rightarrow M_1 \otimes_R M_2 \rightarrow 0$$

where J is generated by the extra relations imposed by R -bilinearity (beyond W -bilinearity). For $r \in R$, define $\mu_r : M_1 \otimes M_2 \rightarrow M_1 \otimes M_2$ by

$$\mu_r(m_1 \otimes m_2) = (rm_1) \otimes m_2 - m_1 \otimes (rm_2).$$

Then $J = \sum_{r \in R} \mathrm{Im}(\mu_r)$; this is an object the abelian category $\mathrm{MF}_{W,\mathrm{tor}}^{f,[a,b]}$. We will show that $T_{\mathrm{cris}}(J)$ is the kernel of $T_{\mathrm{cris}}(M_1 \otimes M_2) \rightarrow T_{\mathrm{cris}}(M_1) \otimes_R T_{\mathrm{cris}}(M_2)$.

It suffices to show that $T_{\mathrm{cris}}(N_1 + N_2) = T_{\mathrm{cris}}(N_1) + T_{\mathrm{cris}}(N_2)$ for subobjects N_1 and N_2 of $M_1 \otimes M_2$. Indeed, granting this we would know that

$$T_{\mathrm{cris}}(J) = \sum_{r \in R} T_{\mathrm{cris}}(\mu_r).$$

But by functoriality $T_{\mathrm{cris}}(\mu_r)$ is the map $T_{\mathrm{cris}}(M_1) \otimes T_{\mathrm{cris}}(M_2) \rightarrow T_{\mathrm{cris}}(M_1) \otimes T_{\mathrm{cris}}(M_2)$ given by $v_1 \otimes v_2 \mapsto rv_1 \otimes v_2 - v_1 \otimes rv_2$, so $T_{\mathrm{cris}}(J)$ is the kernel of $T_{\mathrm{cris}}(M_1 \otimes M_2) \rightarrow T_{\mathrm{cris}}(M_1) \otimes_R T_{\mathrm{cris}}(M_2)$ as desired.

To prove that $T_{\text{cris}}(N_1 + N_2) = T_{\text{cris}}(N_1) + T_{\text{cris}}(N_2)$, consider the exact sequence

$$0 \rightarrow N_1 \cap N_2 \rightarrow N_1 \oplus N_2 \rightarrow N_1 + N_2 \rightarrow 0.$$

As T_{cris} preserves direct sums, it suffices to show that

$$T_{\text{cris}}(N_1) \cap T_{\text{cris}}(N_2) = T_{\text{cris}}(N_1 \cap N_2).$$

But this follows from the exactness of T_{cris} and the left exact sequence

$$0 \rightarrow N_1 \cap N_2 \rightarrow N_1 \oplus N_2 \rightarrow M_1 \otimes M_2$$

where the second map is $(n_1, n_2) \mapsto n_1 - n_2$. \square

8.3. Duality. Let R be a coefficient ring over \mathcal{O} and $M \in \text{MF}_{W, \text{tor}}^f$ have the structure of a free R -module. Fix $L \in \text{MF}_{W, \text{tor}}^f$ with an R -structure so that for each τ , L_τ is a free R -module of rank 1 with $L_\tau^{s_\tau} = L_\tau$ and $L_\tau^{s_\tau+1} = 0$ for some s_τ (the analogue of a character taking values in R^\times). We will define a dual relative to L akin to Cartier duality. This will be useful for studying pairings.

Definition 8.20. For an M as above, define $M^\vee = \text{Hom}_{R \otimes_{\mathbb{Z}_p} W}(M, L)$ with a filtration given by

$$\text{Fil}^i M^\vee = \{\psi \in \text{Hom}_{R \otimes_{\mathbb{Z}_p} W}(M, L) : \psi(M^j) \subset L^{i+j} \text{ for all } j \in \mathbb{Z}\}.$$

For $\psi \in \text{Fil}^i M^\vee$, define $\varphi_{M^\vee}^i(\psi)$ to be the unique function in $\text{Hom}_{R \otimes_{\mathbb{Z}_p} W}(M, L)$ such that

$$\varphi_{M^\vee}^i(\psi)(\varphi_M^j(m)) = \varphi_L^{i+j}(\psi(m)).$$

for all $m \in M^j$ and j .

If $\varphi_{M^\vee}^i$ exists, it is unique since the images of the φ_M^j 's span M additively. Likewise, if $\varphi_{M^\vee}^i$ exists for all i they are automatically σ -semilinear and satisfy $p\varphi_{M^\vee}^{i+1} = \varphi_{M^\vee}^i|_{\text{Fil}^{i+1} M^\vee}$. We check $\varphi_{M^\vee}^i(\psi)$ is well-defined in the following lemma. The key fact is that all of the M_τ^i are free R -module direct summands of M_τ (by Lemma 8.16).

Lemma 8.21. *The function $\varphi_{M^\vee}^i(\psi)$ is well-defined, and the filtration can equivalently be described as*

$$\text{Fil}^i M^\vee = \bigoplus_{\tau: W \hookrightarrow \mathcal{O}} \text{Hom}_R(M_\tau / M_\tau^{1+s_\tau-i}, L_\tau).$$

Proof. We first establish the alternate description of $\text{Fil}^i M^\vee$. Because

$$\text{Hom}_{R \otimes_{\mathbb{Z}_p} W}(M, L) = \bigoplus_{\tau: W \hookrightarrow \mathcal{O}} \text{Hom}_R(M_\tau, L_\tau),$$

and $L_\tau^{s_\tau} = L_\tau$ while $L_\tau^{s_\tau+1} = 0$, an element $\psi_\tau \in \text{Hom}_R(M_\tau, L_\tau)$ satisfies $\psi_\tau(M_\tau^j) \subset L_\tau^{i+j}$ if and only if $\psi_\tau(M_\tau^j) = 0$ whenever $i+j > s_\tau$. This says exactly that ψ_i factors through $M_\tau / M_\tau^{1+s_\tau-i}$. Because $M_\tau / M_\tau^{1+s_\tau-i}$ is an R -module direct summand, hence free with free complement, a morphism $M_\tau / M_\tau^{1+s_\tau-i} \rightarrow L_\tau$ is equivalent to a morphism $\psi_\tau : M_\tau \rightarrow L_\tau$ such that $\psi_\tau(M_\tau^{1+s_\tau-i}) = 0$. Thus $\text{Fil}^i M^\vee = \text{Hom}_R(M_\tau / M_\tau^{1+s_\tau-i}, L_\tau)$ as desired.

We will construct $\varphi_{M^\vee}^i : \text{Fil}^i M^\vee \rightarrow M^\vee$ using the exact sequence

$$(8.1) \quad 0 \rightarrow \bigoplus_{r=a+1}^b M^r \rightarrow \bigoplus_{r=a}^b M^r \rightarrow M \rightarrow 0$$

of [FL82, Lemme 1.7]. The first map sends $(m_r)_{r=a+1}^{r=b}$ to $(pm_r - m_{r+1})_{r=a}^{r=b}$ (with the convention that $m_a = 0$ and $m_{b+1} = 0$), and the second map is $\sum_{r=a}^b \varphi_M^i$. For $\psi \in \text{Fil}^i M^\vee$, consider the map

$$\phi : \bigoplus_{r=a}^b M^r \rightarrow L_\sigma$$

induced by the $\varphi_L^{i+r} \circ \psi : M^r \rightarrow L_\sigma$. For $(m_r)_{r=a+1}^{r=b}$ in $\bigoplus_{r=a+1}^b M^r$, we compute that

$$\begin{aligned} \phi((m_r)_{r=a+1}^{r=b}) &= \sum_{j=a}^b \varphi_L^{i+j}(\psi((pm_j - m_{j+1}))) \\ &= \sum_{j=a}^b p\varphi_L^{i+j}(\psi(m_j)) - \sum_{j=a}^b \varphi_L^{i+j}(\psi(m_{j+1})). \end{aligned}$$

But $\varphi_L^{i+j}|_{L^{i+j+1}} = p\varphi_L^{i+j+1}$, so this difference is

$$\sum_{j=a}^b p\varphi_L^{i+j}(\psi(m_j)) - \sum_{j=a+1}^{b+1} p\varphi_L^{i+j}(\psi(m_j))$$

which vanishes as $m_{b+1} = 0$ and $m_a = 0$. Hence ϕ factors through the quotient M of (8.1), giving the desired well-defined map $\varphi_{M^\vee}^i$. \square

Lemma 8.22. *The Fontaine-Laffaille module M^\vee is an object of $\mathrm{MF}_{W,\mathrm{tor}}^f$.*

Proof. It suffices to show that the inclusion

$$\sum_i \varphi_{M^\vee}^i(\mathrm{Fil}^i M^\vee) \hookrightarrow M^\vee$$

is an equality. By Nakayama's lemma, it suffices to show that the reduction modulo \mathfrak{m}_R is surjective. For an R -module N , let \overline{N} denote the reduction modulo \mathfrak{m}_R . We may pick free R -modules N_τ^i such that $M_\tau^i = N_\tau^i \oplus M_\tau^{i+1}$ as each M_τ^i is a (free) direct summand of the R -free M_τ that is an R -free direct summand of M . Because $p \cdot \varphi_M^{i+1} = \varphi_M^i|_{M^{i+1}}$, we see $\varphi_M^i(\overline{M}_\tau^i) = \varphi_M^i(\overline{N}_\tau^i)$, so

$$\overline{M}_{\sigma\tau} = \sum_i \varphi_M^i(\overline{N}_\tau^i).$$

By Lemma 8.16, \overline{M}_τ and $\overline{M}_{\sigma\tau}$ have the same dimension so $\varphi_M^i|_{N_\tau^i}$ is injective and the sum is direct. We also know that $\varphi_L^i|_{\overline{L}_\tau} = 0$ for $i < s_\tau$ because $p \cdot \varphi_L^{j+1} = \varphi_L^j|_{L^{j+1}}$.

As M_τ and L_τ are free R -module summands of M and L for all τ , $\overline{M}^\vee = \overline{M}^\vee$ by Lemma 8.21. We can describe an element $\psi \in \mathrm{Fil}^i \overline{M}^\vee$ as a collection of $\psi_{\tau,j} \in \bigoplus_{\tau,j} \mathrm{Hom}_R(\overline{N}_\tau^j, \overline{L}_\tau^{i+j})$. But \overline{L}_τ^{i+j} is one-dimensional over k if $i+j \leq s_\tau$, and is zero otherwise. Then for $f = \varphi_{M^\vee}^i(\psi)$ and $m = \sum_{\tau,j} \varphi_M^j(n_{\tau,j})$ with $n_{\tau,j} \in \overline{N}_\tau^j$, by construction we have

$$f(m) = \sum_{\tau,j} \varphi_L^{i+j}(\psi(n_{\tau,j})).$$

But $\varphi_L^{i+j}(\psi(n_{\tau,j}))$ is forced to be zero unless $i+j = s_\tau$, in which case it can take on any non-zero value in \overline{L}_τ (depending on the choice of ψ). Thus

$$\varphi_{M^\vee}^i(\mathrm{Fil}^i \overline{M}^\vee) = \bigoplus_\tau \mathrm{Hom}(\varphi_M^{s_\tau-i}(\overline{N}_\tau^{s_\tau-i}), \overline{L}_{\sigma\tau}).$$

Summing over i , and using the sum decomposition $\overline{M} = \sum_{\tau,i} \varphi_M^i(\overline{N}_\tau^i)$ gives that

$$\sum_i \varphi_{M^\vee}^i(\mathrm{Fil}^i \overline{M}^\vee) = \mathrm{Hom}(\overline{M}, \overline{L}).$$

This shows the desired surjectivity. \square

Remark 8.23. For fixed \mathbf{Z}_p -embedding $\tau : W \hookrightarrow \mathcal{O}$, if the Fontaine-Laffaille weights (Remark 8.17) of M with respect to τ are $\{w_{\tau,i}\}_i$ then the Fontaine-Laffaille weights of M^\vee with respect to τ are $\{s_\tau - w_{\tau,i}\}_i$.

Letting ν be the Galois representation on the free rank-1 R -module corresponding to L , we define the dual $V^\vee = \text{Hom}_{R[\Gamma_K]}(V, R(\nu))$ for a discrete Γ_K -representation on a finite free R -module V .

Lemma 8.24. *For a morphism $f : M \rightarrow N$ in $\text{MF}_{W,\text{tor}}^{f,[a,b]}$ with $b - a \leq \frac{p-2}{2}$, there is a natural isomorphism $T_{\text{cris}}(M^\vee) \simeq T_{\text{cris}}(M)^\vee$ and $T_{\text{cris}}(f^\vee) = T_{\text{cris}}(f)^\vee$.*

Proof. We prove this by studying the evaluation pairing $M \otimes_R M^\vee \rightarrow L$. It is straightforward to verify that this pairing is a morphism of Fontaine-Laffaille modules. Because $b - a \leq \frac{p-2}{2}$, Lemma 8.19 gives a pairing of Galois-modules

$$(8.2) \quad T_{\text{cris}}(M) \otimes_R T_{\text{cris}}(M^\vee) = T_{\text{cris}}(M \otimes_R M^\vee) \rightarrow T_{\text{cris}}(L) = \nu_R.$$

We will now prove that this pairing is perfect when $R = k$. We will do so by inducting on the dimension of the k -vector space M . The case of dimension 0 is clear. Also, if $M \neq 0$ the pairing of Fontaine-Laffaille modules is non-zero (look at the pairing $M_\tau \times \text{Hom}(M_\tau, L_\tau) \rightarrow L_\tau$ of vector spaces). Thus the pairing of Galois-modules is non-zero if $M \neq 0$ as T_{cris} is faithful.

Now we use induction, so we can assume $M \neq 0$. The annihilator of $T_{\text{cris}}(M^\vee)$ is $T_{\text{cris}}(M_1)$ for some $f : M_1 \hookrightarrow M$ because the essential image of T_{cris} is closed under taking sub-objects. We know M_1 is a proper sub-object as the pairing is non-zero. Observe that we may define the dual $f^\vee : M^\vee \rightarrow M_1^\vee$ by precomposition: it is surjective as we are over a field. For $v_1 \in T_{\text{cris}}(M_1)$ and $v_2 \in T_{\text{cris}}(M^\vee)$, we must have

$$0 = \langle v_1, f^\vee v_2 \rangle = \langle f(v_1), v_2 \rangle.$$

But the pairing $T_{\text{cris}}(M_1) \otimes T_{\text{cris}}(M_1^\vee) \rightarrow T_{\text{cris}}(L)$ is non-degenerate by induction, and f^\vee is surjective, so this means that $v_1 = 0$. Thus $T(M_1)$ and hence M_1 are trivial. Over the field k , this ensures the pairing is perfect.

For the general case, we use the basic fact that for a coefficient ring R , if N_1 and N_2 are free R -modules of the same rank with an R -bilinear pairing $N_1 \times N_2 \rightarrow R$, the pairing is perfect if the reduction (modulo \mathfrak{m}_R) $\overline{N}_1 \times \overline{N}_2 \rightarrow k$ is perfect. Apply this to $T_{\text{cris}}(M) \times T_{\text{cris}}(M^\vee) \rightarrow T_{\text{cris}}(L)$.

The statement $T_{\text{cris}}(f^\vee) = T_{\text{cris}}(f)^\vee$ is just functoriality. \square

9. FONTAINE-LAFFAILLE DEFORMATIONS

Let $G = \text{GSp}_r$ or GO_r , and consider a representation $\overline{\rho} : \Gamma_K \rightarrow G(k)$ with similitude character $\overline{\nu}$, where $K = W[\frac{1}{p}]$ for $W = W(k')$ with finite k' . Let \overline{V} be the underlying vector space for $\overline{\rho}$ using the standard representation of G . Take \mathcal{O} to be the Witt vectors of k , and assume $\mathcal{O}[\frac{1}{p}]$ splits K over \mathbf{Q}_p . Fix a lift $\nu : \Gamma_K \rightarrow \mathcal{O}^\times$ of $\overline{\nu}$ that is crystalline with Hodge-Tate weights $\{s_\tau\}_\tau$ in an interval of length $p - 2$, corresponding to a filtered Dieudonné module L .

We suppose that $\overline{\rho}$ is torsion-crystalline with Hodge-Tate weights in an interval $[a, b]$ where $0 \leq b - a \leq \frac{p-2}{2}$ so we can use Fontaine-Laffaille theory. Let \overline{M} be the corresponding Fontaine-Laffaille module (using Fact 8.10(6)), with Fontaine-Laffaille weights $\{w_{\tau,i}\}_{\tau,i}$. In this section we define and study the Fontaine-Laffaille deformation condition assuming that for each \mathbf{Z}_p -embedding $\tau : W \hookrightarrow \mathcal{O}$ the Fontaine-Laffaille weights are multiplicity-free as in Remark 8.17 (the jumps in the filtration are of rank 1).

9.1. Definitions and Basic Properties. As \overline{V} is an k -linear representation of Γ_K , \overline{M} becomes a $k' \otimes_{\mathbf{F}_p} k$ -module and in particular a k -vector space.

Definition 9.1. For a coefficient ring R over $\mathcal{O} = W(k)$, define $D_{\overline{\rho}}^{\text{FL}}(R)$ to be the collection of lifts $\rho : \Gamma_K \rightarrow G(R)$ of $\overline{\rho}$ with similitude character ν_R that lie in the essential image of T_{cris} restricted to the full subcategory $\text{MF}_{W,\text{tor}}^{f,[a,b]}$. Such a deformation is called a *Fontaine-Laffaille deformation*.

We will analyze this deformation condition when for each fixed embedding $\tau : W \hookrightarrow \mathcal{O}$ the Fontaine-Laffaille weights of $\bar{\rho}$ are multiplicity-free (when the jumps in the filtration of each \overline{M}_τ are 1-dimensional over k).

Theorem 9.2. *If the Fontaine-Laffaille weights are multiplicity-free, $D_{\bar{\rho}}^{\text{FL}}$ is liftable. If B is a Borel subgroup of G , the dimension of the tangent space of $D_{\bar{\rho}}^{\text{FL}}$ is*

$$[K : \mathbf{Q}_p] (\dim G_k - \dim B_k) + H^0(\Gamma_K, \text{ad}^0(\bar{\rho})).$$

If $\rho : \Gamma_K \rightarrow G(\mathcal{O})$ is an inverse limit of Fontaine-Laffaille deformations of $\bar{\rho}$ to $\mathcal{O}/p^n\mathcal{O}$ for all $n \geq 1$, it is a lattice in a crystalline representation with the same Fontaine-Laffaille weights as $\bar{\rho}$.

The proof of this theorem will occur over the remainder of this section. The key pieces are Proposition 9.7, Proposition 9.8, and Proposition 9.20.

To understand $D_{\bar{\rho}}^{\text{FL}}$, we must express the orthogonal or symplectic pairing in the language of Fontaine-Laffaille modules. For a Galois module V which is a free R -module, recall we defined $V^\vee = \text{Hom}_{R[\Gamma_K]}(V, \nu_R)$. For a deformation of $\bar{\rho}$ to a coefficient ring R , we obtain an $R[\Gamma_K]$ -module V together with an isomorphism $\eta : V \simeq V^\vee$ coming from the pairing. Let $\epsilon = 1$ for $G = \text{GO}_r$ and $\epsilon = -1$ for $G = \text{GSp}_r$. The fact that $\langle v, w \rangle = \epsilon \langle w, v \rangle$ is equivalent to $\eta^* = \epsilon \eta$, where η^* is the map $V \simeq V^{\vee\vee} \rightarrow V^\vee$ induced by double duality.

Lemma 9.3. *For a coefficient ring R , suppose V is a lift of \bar{V} as an $R[\Gamma_K]$ -module that is finite free over R , corresponding to a Fontaine-Laffaille module M that is finite free over R . An isomorphism of $R[\Gamma_K]$ -modules*

$$\eta : V \simeq V^\vee$$

such that $\eta(v)(w) = \epsilon \eta(w)(v)$ is equivalent to an R -linear isomorphism of Fontaine-Laffaille modules

$$\gamma : M \simeq M^\vee$$

such that $\gamma(m)(n) = \epsilon \gamma(n)(m)$.

Proof. As the Hodge-Tate weights of $\bar{\rho}$ lie in an interval of length $\frac{p-2}{2}$, Lemma 8.19 and Lemma 8.24 hold. In particular, $T_{\text{cris}}(M^\vee) = T_{\text{cris}}(M)^\vee$. As T_{cris} is fully faithful in this range, we see that a map η is equivalent to a map γ , and one is an isomorphism if and only if the other one is. It remains to check that γ is symmetric or alternating if and only if η is. Let η^* and γ^* denote the isomorphisms respectively given by

$$V \simeq V^{\vee\vee} \xrightarrow{\eta^\vee} V^\vee \quad \text{and} \quad M \simeq M^{\vee\vee} \xrightarrow{\gamma^\vee} M^\vee.$$

A straightforward check shows that T_{cris} carries η^* to γ^* , and hence $\eta = \epsilon \eta^*$ if and only if $\gamma = \epsilon \gamma^*$. \square

Lemma 9.4. *An R -linear isomorphism of Fontaine-Laffaille modules $\gamma : M \simeq M^\vee$ for which $\gamma(m)(n) = \epsilon \gamma(n)(m)$ is equivalent to a perfect ϵ -symmetric R -bilinear pairing $\langle \cdot, \cdot \rangle : M \times M \rightarrow L_R$ satisfying*

- $\langle M^i, M^j \rangle \subset L^{i+j}$;
- $\langle \varphi_M^i(m), \varphi_M^j(n) \rangle = \varphi_L^{i+j} \langle m, n \rangle$.

Proof. This is just writing out what $\gamma : M \rightarrow M^\vee$ being a morphism of Fontaine-Laffaille modules means for the pairing $\langle m, n \rangle = \gamma(m)(n)$.

For γ to preserve the filtration says exactly that

$$\gamma(M^i) \subset \text{Fil}^i M^\vee = \{\psi \in \text{Hom}_R(M, L) : \psi(M^j) \subset L^{i+j}\}.$$

This is equivalent to $\langle M^i, M^j \rangle \subset L^{i+j}$ for all i, j . The compatibility of γ with the φ 's says exactly that for $m \in M^i$

$$\varphi_{M^\vee}^i(\gamma(m)) = \gamma(\varphi_M^i(m)).$$

Evaluating on any $\varphi_M^j(n) \in M$ and using the definition of M^\vee we see

$$\varphi_{M^\vee}^i(\gamma(m))(\varphi_M^j(n)) = \varphi_L^{i+j}(\gamma(m)(n)) = \varphi_L^{i+j}(\langle m, n \rangle).$$

Evaluating $\gamma(\varphi_M^i(m))$, we see that

$$\gamma(\varphi_M^i(m))(\varphi_M^j(n)) = \langle \varphi_M^i(m), \varphi_M^j(n) \rangle.$$

Thus, γ being compatible with the φ 's is equivalent to $\langle \varphi_M^i(m), \varphi_M^j(n) \rangle = \varphi_L^{i+j}(\langle m, n \rangle)$. \square

In particular, the pairing $\overline{V} \times \overline{V} \rightarrow \overline{v}$ gives a perfect pairing $\langle \cdot, \cdot \rangle_{\overline{M}} : \overline{M} \times \overline{M} \rightarrow \overline{L}$.

Corollary 9.5. *For a coefficient ring R , a lift $\rho \in D_{\overline{\rho}}^{\text{FL}}(R)$ is equivalent to a Fontaine-Laffaille module $M \in \text{MF}_{W, \text{tor}}^{f, [a, b]}$ that is free as an R -module for which there exists a perfect ϵ -symmetric R -bilinear pairing $\langle \cdot, \cdot \rangle : M \times M \rightarrow L_R$ satisfying*

- $\langle M^i, M^j \rangle \subset L^{i+j}$;
- $\langle \varphi_M^i(m), \varphi_M^j(n) \rangle = \varphi_L^{i+j}(\langle m, n \rangle)$.

such that $(M, \langle \cdot, \cdot \rangle)$ reduces to $(\overline{M}, \langle \cdot, \cdot \rangle_{\overline{M}})$.

Proof. This follows by combining the two previous lemmas. Note that the pairing $\langle \cdot, \cdot \rangle$ is automatically perfect as it lifts the perfect pairing $\langle \cdot, \cdot \rangle_{\overline{M}}$. \square

Corollary 9.6. $D_{\overline{\rho}}^{\text{FL}}$ is a deformation condition.

Proof. This follows from the fact that for a morphism of coefficient rings $R \rightarrow R'$, $R' \otimes_R T_{\text{cris}}(M) = T_{\text{cris}}(R' \otimes_R M)$, exactness properties of T_{cris} on $\text{MF}_{W, \text{tor}}^f$, and Corollary 9.5. For example, to check that $D_{\overline{\rho}}^{\text{FL}}$ is a sub-functor of $\mathcal{D}_{\overline{\rho}}$, let R be a coefficient ring and M be the Fontaine-Laffaille module corresponding to $\rho \in D_{\overline{\rho}}^{\text{FL}}(R)$. Then $R' \otimes_R T_{\text{cris}}(M)$ lies in the essential image of T_{cris} , and $R' \otimes_R M$ admits a perfect ϵ -symmetric R' -bilinear pairing as in Corollary 9.5 given by extending the pairing on M . This shows that $\rho_{R'} \in D_{\overline{\rho}}^{\text{FL}}(R')$. A similar argument checks Definition 2.5(2). \square

It is simple to understand characteristic-zero points of the deformation functor.

Proposition 9.7. *Suppose we are given a compatible collection of Fontaine-Laffaille deformations $\rho_i : \Gamma_K \rightarrow G(R_i)$, where $\{R_i\}$ is a co-final system of artinian quotients of the valuation ring R of a finite extension of $\mathcal{O}[\frac{1}{p}]$ with the same residue field as \mathcal{O} . Then $\rho = \varprojlim \rho_i$ is crystalline (more precisely, a lattice in a crystalline representation) with indexed tuple of Hodge-Tate weights equal to the corresponding indexed-tuple of Fontaine-Laffaille weights of $\overline{\rho}$.*

Proof. It is straightforward to verify that the inverse limit of the Fontaine-Laffaille modules corresponding to ρ_i is in \mathcal{D}_K . Then the result follows from combining Fact 8.10(2) and (5). Our convention that the cyclotomic character has Hodge-Tate weight -1 makes the Hodge-Tate weights and Fontaine-Laffaille weights match (Remark 8.11). \square

9.2. Liftability. In this section, we analyze liftability by constructing lifts of Fontaine-Laffaille modules. Lifting the underlying module, filtration, and pairing will be relatively easy. Constructing lifts of the φ_M^i compatible with these choices requires substantial work. Let $\mathcal{W}_{\text{FL}, \tau}$ denote the Fontaine-Laffaille weights of $\overline{\rho}$ with respect to a \mathbf{Z}_p -embedding $\tau : W \hookrightarrow \mathcal{O}$, corresponding to the jumps in the filtration of \overline{M}_τ .

Proposition 9.8. *Under the assumption that the Fontaine-Laffaille weights lie in an interval of length $\frac{p-2}{2}$ and are multiplicity-free for each $\tau : W \hookrightarrow \mathcal{O}$, the deformation condition $D_{\overline{\rho}}^{\text{FL}}$ is liftable.*

Let $\rho : \Gamma_K \rightarrow G(R)$ be a Fontaine-Laffaille deformation of $\bar{\rho}$. Let M and \overline{M} be the corresponding Fontaine-Laffaille modules for ρ and $\bar{\rho}$, which decompose as

$$M = \bigoplus_{\tau} M_{\tau} \quad \text{and} \quad \overline{M} = \bigoplus_{\tau} \overline{M}_{\tau}.$$

Each M_{τ} is a free R -module by Lemma 8.16. Furthermore, the filtration $\{M_{\tau}^i\}$ on M_{τ} is given by R -module direct summands and $\varphi_M^i(M_{\tau}^i) \subset M_{\sigma\tau}$. In particular, there exist free rank-1 R -modules $N_{\tau}^i \subset M_{\tau}^i$ such that $M_{\tau}^i = N_{\tau}^i \oplus M_{\tau}^{i+1}$. As the pairing is \mathcal{O} -bilinear, the pairings $M_{\tau} \times M_{\tau} \rightarrow L_{\tau}$ are collectively equivalent to the pairing $M \times M \rightarrow L$, so to lift the pairing and check compatibility it suffices to do so on M_{τ} . Thus to analyze liftability of M , we will work with each M_{τ} separately using $R \otimes_{\mathbf{Z}_p} W = \prod_{\tau} R_{\tau}$ with τ varying through \mathbf{Z}_p -embeddings $W \hookrightarrow \mathcal{O} \rightarrow R$.

By a *basis* for M_{τ} , we mean a basis for it as an R -module. By Lemma 8.16, the rank of M_{τ} is r . For $G = \mathrm{GSp}_r$ with r even, the *standard alternating pairing* with respect to a chosen basis is the one given by the block matrix

$$\begin{pmatrix} 0 & I'_{r/2} \\ -I'_{r/2} & 0 \end{pmatrix}$$

where I'_m denotes the anti-diagonal matrix with 1's on the diagonal. For $G = \mathrm{GO}_r$, the *standard symmetric pairing* with respect to the basis is the one given by the matrix I'_r .

Example 9.9. Take $R = k$ and fix an embedding $\tau : W \hookrightarrow \mathcal{O}$. Let w_1, \dots, w_r be the Fontaine-Laffaille weights of M_{τ} , and recall that $w_i + w_{r+1-i} = s_{\tau}$ because $M \simeq M^{\vee}$. Pick $v_i \in M_{\tau}^{w_i} - M_{\tau}^{w_i+1}$. Since $\varphi_M^i|_{M_{i+1}} = p\varphi_M^{i+1} = 0$,

$$M_{\sigma\tau} = \sum_i \varphi^i(M_{\tau}^i) = \mathrm{span}_k \varphi_M^{w_i}(v_i).$$

Note that $\{\varphi_M^{w_i}(v_i)\}$ is a k -basis for $M_{\sigma\tau}$, as the left side has k -dimension r and there are r Fontaine-Laffaille weights for τ . Furthermore, compatibility with the pairing means that

$$\langle \varphi_M^{w_i}(v_i), \varphi_M^{w_j}(v_j) \rangle = \varphi_L^{w_i+w_j}(\langle v_i, v_j \rangle).$$

But $\varphi_L^h|_{L_{\tau}} = 0$ unless $h = s_{\tau}$: for $h > s_{\tau}$ this is because $L_{\tau}^h = 0$, while for $h < s_{\tau}$ this is because $L_{\tau}^h = L_{\tau}^{h+1} = L_{\tau}$ and $\varphi_L^h|_{L_{\tau}^{h+1}} = p\varphi_L^{h+1} = 0$. Thus $\langle \varphi_M^{w_i}(v_i), \varphi_M^{w_j}(v_j) \rangle = 0$ unless $w_i + w_j = s_{\tau}$, in which case the pairing must be non-zero as it is perfect. If $i \neq j$, by rescaling v_i we may arrange for $\langle \varphi_M^{w_i}(v_i), \varphi_M^{w_j}(v_j) \rangle$ to be an arbitrary unit. For $G = \mathrm{GSp}_r$ or $G = \mathrm{GO}_r$ with r even this means after rescaling the pairing may be taken to be standard with respect to the basis $n_i = \varphi^{w_i}(v_i)$ of $M_{\sigma\tau}$. For $G = \mathrm{GO}_r$ with r odd and $i = [r/2] + 1$, defining $\omega_{\tau} := \langle \varphi^{w_i}(v_i), \varphi^{w_i}(v_i) \rangle \in k^{\times}$ and rescaling v_1, \dots, v_{i-1} then brings us to the case that the pairing is ω_{τ} times the standard pairing with respect to the basis $n_i = \varphi^{w_i}(v_i)$ of $M_{\sigma\tau}$.

Remark 9.10. The constant ω_{τ} depends on the choice of basis $\{v_i\}$ for M_{τ} , so in particular is not independent of τ . This will not cause problems in later arguments.

Remark 9.11. There is a lot of notation in the following arguments. With τ fixed, we will use v_i to denote elements of $M_{\tau}^{w_i}$, and m_i to denote elements of $M_{\sigma\tau}$. Usually we will have $\varphi_M^{w_i}(v_i) = m_i$. If we want to index by Fontaine-Laffaille weights instead of the integers $\{1, 2, \dots, r\}$, we will use $v'_{w_i} := v_i$ and $m'_{w_i} := m_i$.

Lemma 9.12. Let $w_1 < w_2 < \dots < w_r$ denote the Fontaine-Laffaille weights of M with respect to τ . There exists an R -basis m_1, \dots, m_r of $M_{\sigma\tau}$ such that $m_i = \varphi_M^{w_i}(v_i)$ where v_i is an R -basis for a complement to $M_{\tau}^{w_i+1}$ in $M_{\tau}^{w_i}$ and such that the pairing $\langle \cdot, \cdot \rangle$ on $M_{\sigma\tau}$ is an R^{\times} -multiple of the standard pairing with respect to the basis $\{m_i\}$.

Proof. Example 9.9 shows that such a basis \bar{v}_i exists over R/\mathfrak{m}_R : pick a lift $v_i \in N_\tau^i$ of \bar{v}_i , and define $m_i = \varphi_M^{w_i}(v_i)$. We know that

$$\langle \varphi_M^{w_i} v_i, \varphi_M^{w_j} v_j \rangle = \varphi_L^{w_i+w_j}(\langle v_i, v_j \rangle).$$

If $w_i + w_j > s_\tau$, this is zero because $L_\tau^{s_\tau+1} = 0$. If $w_i + w_j < s_\tau$, since $\varphi_L^{w_i+w_j}|_{L_\tau^{s_\tau}} = p^{s_\tau-w_i-w_j} \varphi_L^{s_\tau}$ this is not a unit. If $w_i + w_j = s_\tau$ (equivalently, $i + j = r + 1$), it is a unit of R as the pairing is perfect.

We will modify the lifts v_i and then take $m_i = \varphi_M^{w_i}(v_i)$. For $0 \leq j \leq r/2$ (so $j < r + 1 - j$), we will inductively arrange that:

- (1) for $i \leq j$, $\langle m_i, m_h \rangle = 0$ for $h \neq r + 1 - i$;
- (2) v_i is an R -basis for a complement to $M_\tau^{w_i+1}$ in $M_\tau^{w_i}$;
- (3) $\langle m_i, m_{r+1-i} \rangle$ is a unit for all $1 \leq i \leq r$.

For $j = 0$, the first condition is vacuous and the other two conditions hold by our choice of lift. Given that these conditions hold for $j - 1$ with $1 \leq j \leq \frac{r}{2}$, we will show how to modify the v_i so that these conditions hold for j . Let $c = \langle m_j, m_{r+1-j} \rangle \in R^\times$. For $j < h < r + 1 - j$, define

$$\tilde{v}_h := v_h - \langle m_j, m_h \rangle c^{-1} v_{r+1-j}.$$

As $j \neq r + 1 - h$, $\langle m_j, m_h \rangle \in \mathfrak{m}_R$ so \tilde{v}_h lifts \bar{v}_h . We compute that

$$\langle m_j, \varphi_M^{w_h} \tilde{v}_h \rangle = \langle m_j, m_h \rangle - \langle m_j, m_h \rangle c^{-1} \langle m_j, m_{r+1-j} \rangle = 0.$$

For $i < j$, as $r + 1 - i \neq h, r + 1 - h$ we know m_i is orthogonal to both m_h and m_{r+1-h} by the inductive hypothesis and hence $\langle m_i, \varphi_M^{w_h} \tilde{v}_h \rangle = 0$. Thus (1) holds for the R -basis

$$v_1, \dots, v_j, \tilde{v}_{j+1}, \dots, \tilde{v}_{r-j}, v_{r-j+1}, \dots, v_r.$$

As $\tilde{v}_h - v_h \in M_\tau^{w_{r+1-j}}$, \tilde{v}_h is still an R -basis for a complement to $M_\tau^{w_h+1}$ in $M_\tau^{w_h}$ (since $w_{r+1-j} > w_h$ as $h < r + 1 - j$), so (2) holds for this new R -basis of M_τ . Furthermore, we see that

$$\langle \varphi_M^{w_h} \tilde{v}_h, \varphi_M^{w_{r+1-h}} \tilde{v}_{r+1-h} \rangle - \langle m_h, m_{r+1-h} \rangle \in \mathfrak{m}_R.$$

As $\langle m_h, m_{r+1-h} \rangle$ is a unit, $\langle \varphi_M^{w_h} \tilde{v}_h, \varphi_M^{w_{r+1-h}} \tilde{v}_{r+1-h} \rangle$ is a unit and (3) holds. Thus we may modify the lifts v_i and then accordingly modify m_i to satisfy the inductive hypothesis.

Take such a basis for $j = \lfloor r/2 \rfloor$. By (1),

$$\langle m_i, m_{i'} \rangle = 0$$

if $i + i' \neq r + 1$ and one of i or i' is at most $r/2$. Otherwise $i' > r + 1 - i$ so $w_i + w_{i'} > s_\tau$ and hence the pairing is zero automatically. If r is even, rescale $v_1, \dots, v_{r/2}$ so that $\langle m_i, m_{r+1-i} \rangle = 1$ for $i \leq r/2$ using (3). If r is odd (so $G = \text{GO}_r$), let $\omega_\tau = \langle v_{\lfloor r/2 \rfloor + 1}, v_{\lfloor r/2 \rfloor + 1} \rangle \in R^\times$ and rescale $v_1, \dots, v_{\lfloor r/2 \rfloor}$ so that $\langle m_i, m_{r+1-i} \rangle = \omega_\tau$ for $1 \leq i \leq \lfloor r/2 \rfloor$. In these cases, the pairing with respect to the basis v_1, \dots, v_r is a multiple of the standard pairing. \square

Remark 9.13. When r is odd (so $G = \text{GO}_r$), to choose a basis where the pairing is standard we would need to rescale $v_{\lfloor r/2 \rfloor + 1}$ by a square root of the unit $\langle m_{\lfloor r/2 \rfloor + 1}, m_{\lfloor r/2 \rfloor + 1} \rangle$. This might not exist in R . But note that the orthogonal similitude group GO_r is unaffected by a unit scaling of the quadratic form.

Now we begin the proof of Proposition 9.8. Let $R' \twoheadrightarrow R$ be a small surjection with kernel I . To lift ρ to $\rho' : \Gamma_K \rightarrow G(R')$, we can reduce to the case when I is killed by \mathfrak{m}_R and $\dim_k I = 1$. Lift the R -module M_τ together with its pairing $\langle \cdot, \cdot \rangle$ over R' as follows. Choose the basis $\{m_i\}$ provided by Lemma 9.12, with respect to which $\langle \cdot, \cdot \rangle$ is ω_τ times the standard pairing for some $\omega_\tau \in R^\times$. We take M'_{σ_τ} to be a free R' -module with basis $\{n_i\}$ reducing to the basis $\{m_i\}$ of M_{σ_τ} . Lift ω_τ

to some $\omega'_\tau \in (R')^\times$ and define a pairing on M'_τ to be ω'_τ times the standard pairing on M'_τ with respect to $\{n_i\}$. Pick a lift $u_i \in M'_\tau$ of v_i , and define a filtration on M'_τ by

$$(M'_\tau)^j = \text{span}_{R'}(u_i : w_i \geq j).$$

We define the module $M' = \bigoplus_{\tau: W \hookrightarrow \mathcal{O}} M'_\tau$ over $W \otimes_{\mathbf{Z}_p} R$ with filtration $(M')^i = \bigoplus_{\tau: W \hookrightarrow \mathcal{O}} (M'_\tau)^i$. It is clear the filtration reduces to the filtration on M . Furthermore, the pairing $M'_\tau \times M'_\tau \rightarrow L_\tau$ with respect to $\{n_i\}$ is a multiple of the standard one.

It remains to produce $\varphi_{M'}^i$ lifting φ_M^i . As always, it suffices to lift all of the $\varphi_{M'_\tau}^i : M'_\tau \rightarrow M_{\sigma\tau}$ separately. We note that the $\varphi_{M'_\tau}^j : M'^j_\tau \rightarrow M'_{\sigma\tau}$ are determined by the function $\varphi_{M'_\tau}^{j+1}$ on M'^{j+1}_τ (via the relation $p\varphi_{M'_\tau}^{j+1} = \varphi_{M'_\tau}^j|_{M'^{w_j+1}_\tau}$) together with the values $\varphi_{M'_\tau}^{w_i}(u_i)$ for $w_i \in \mathcal{W}_{\text{FL},\tau}$. We will define $\varphi_{M'_\tau}^{w_i}(u_i)$ for each $w_i \in \mathcal{W}_{\text{FL},\tau}$ to obtain the desired set of maps $\varphi_{M'}^j : M'^j \rightarrow M'$.

It will now be more convenient to index via weights, so let $n'_{w_i} = n_i$ and $u'_{w_i} = u_i$. Let us consider defining

$$\varphi_{M'_\tau}^w(u'_w) = \sum_{i \in \mathcal{W}_{\text{FL},\sigma\tau}} c_{iw} n'_i := x_w$$

for c_{iw} to be determined with the obvious restriction that c_{iw} must lift the corresponding coefficient for $\varphi_M^w(v'_w)$. We will study for which choices of $\{c_{iw}\}$ these maps are compatible with the pairing.

Lemma 9.14. *For any choice of $\{c_{iw}\}$, the elements x_w form a basis for $M'_{\sigma\tau}$.*

Proof. Note that the Fontaine-Laffaille weights of \overline{M} , M , and M' are the same. Consider the map

$$\sum_{i \in \mathcal{W}_{\text{FL},\tau}} \varphi_{M'_\tau}^i : M'^i_\tau \rightarrow M'_{\sigma\tau}.$$

Quotienting by the maximal ideal of R' , as φ_M^w is a lift of $\varphi_{\overline{M}}^w$ we obtain a surjection

$$\sum_{i \in \mathcal{W}_{\text{FL},\tau}} \varphi_{\overline{M}}^i : \overline{M}_\tau^i \twoheadrightarrow \overline{M}_{\sigma\tau}$$

as $\overline{M}_{\sigma\tau} = \sum_i \varphi_{\overline{M}}^i(\overline{M}_\tau^i)$. By Nakayama's lemma, the original map is also a surjection. Thus $\{x_w\}$ spans the free R' -module $M'_{\sigma\tau}$. But $\#\{x_w\} = \text{rk}_{R'}(M'_{\sigma\tau}) = r$, so $\{x_w\}$ is a basis for $M'_{\sigma\tau}$. \square

The compatibility condition with the pairing is that

$$\langle \varphi_{M'_\tau}^i(x), \varphi_{M'_\tau}^j(y) \rangle = \varphi_{L_\tau}^{i+j}(\langle x, y \rangle).$$

Let $\epsilon = 1$ for GO_r and $\epsilon = -1$ for GSp_r with even r . Recall that for a Fontaine-Laffaille weight $i \in \mathcal{W}_{\text{FL},\tau}$, we defined i^* to satisfy $i + i^* = s_\tau$, so n'_i and n'_{i^*} pair non-trivially. By linearity and the relation $\langle x, y \rangle = \epsilon \langle y, x \rangle$, it suffices to check compatibility with the pairing only when $i, j \in \mathcal{W}_{\text{FL}}$, $x = n'_i$ and $y = n'_j$ and $i < j$ or $i = j = i^*$ (provided we have arranged that $p\varphi_{M'}^{w+1} = \varphi_{M'}^w|_{M'^{w+1}}$).

Remark 9.15. The case $i = j = i^*$ only occurs when the pairing is orthogonal and r is odd, for the weight of the unique basis vector which pairs with itself giving a unit.

Of course, there is no reason to expect our initial arbitrary choice of $\{c_{iw}\}$ to work. Any other choice is of the form $\{c_{iw} + \delta_{iw}\}$ where $\delta_{iw} \in I$. The compatibility condition on M'_τ becomes

$$\sum_{w, w' \in \mathcal{W}_{\text{FL},\tau}} (c_{iw} + \delta_{iw})(c_{jw'} + \delta_{jw'}) \langle n'_w, n'_{w'} \rangle = \varphi_{L_\tau}^{i+j}(\langle n'_i, n'_j \rangle).$$

Expanding and using the fact that $I^2 = 0$, we see that we wish to choose $\{\delta_{iw}\}$ so that

$$\sum_{w, w' \in \mathcal{W}_{\text{FL},\tau} } (c_{iw}\delta_{jw'} + c_{jw'}\delta_{iw}) \langle n'_w, n'_{w'} \rangle = \omega'_\tau C_{ij}$$

where the constant $C_{ij} := (\omega'_\tau)^{-1} \left(\varphi_{L_\tau}^{i+j}(n'_i, n'_j) - \sum_{w, w' \in \mathcal{W}_{\text{FL}, \tau}} c_{iw} c_{jw'} \langle n'_w, n'_{w'} \rangle \right)$ lies in I as φ_M^i is compatible with the pairing.

Now we can simplify based on the explicit form of the pairing with respect to the basis $\{n'_w\}$. As n'_w only pairs non-trivially with n'_{w^*} , for $i < j$ or $i = j = i^*$ we obtain the relation

$$(9.1) \quad \sum_{w \leq w^*} (c_{iw} \delta_{jw^*} + c_{jw^*} \delta_{iw}) + \epsilon \sum_{w > w^*} (c_{iw} \delta_{jw^*} + c_{jw^*} \delta_{iw}) = C_{ij}.$$

To show that this system of linear equations has a solution, we shall interpret it as a linear transformation.

It is now convenient to index the weights using $\{1, 2, 3, \dots, r\}$. Recall that the Fontaine-Laffaille weights of M_τ are denoted $w_1 < w_2 < \dots < w_r$. Let $U = I^{\oplus r^2}$, and decompose U as $\bigoplus_{i=1}^r U_i$, where the coordinates of $U_i = I^{\oplus r}$ are denoted $\{\delta_{w_i, w_j}\}_{j=1}^r$. Let $U' = I^{\oplus \frac{r(r-1)}{2} + \sigma_r}$, where $\sigma_r = 1$ if there is a $w \in \mathcal{W}_{\text{FL}, \tau}$ for which $w = w^*$ and 0 otherwise. (So σ_r is zero unless $G = \text{GO}_r$ and r is odd.) We may write $U' = \bigoplus_{i=1}^{r-r} U'_i$, where the coordinates of $U'_i = I^{\oplus r-i}$ are denoted $\{C_{w_i w_j}\}_{j=i+1}^r$, except if $\sigma_r = 1$ and $w_i = w_i^*$. In that case, instead take $U'_i = I^{\oplus r-i+1}$ with coordinates denoted $\{C_{w_i w_j}\}_{j=i}^r$.

Consider the function $T : U \rightarrow U'$ given by

$$(\delta_{w_i w_h})_{ih} \mapsto \left(C_{ij} = \sum_{w_h \leq w_h^*} (c_{w_i w_h} \delta_{w_j w_h^*} + c_{w_j w_h^*} \delta_{w_i w_h}) + \epsilon \sum_{w_h > w_h^*} (c_{w_i w_h} \delta_{w_j w_h^*} + c_{w_j w_h^*} \delta_{w_i w_h}) \right)_{ij}$$

where the $c_{ww'} \in R'$ matter only through their images in k since $\mathfrak{m}_R I = 0$. It suffices to show that T is surjective. As we arranged for I to be 1-dimensional over $R/\mathfrak{m}_R = k$, this is question of linear algebra over k upon fixing a k -basis of I .

We will studying particular k -linear maps $U_i \rightarrow U'_i$. To simplify notation, let $\epsilon_i = 1$ except when $w_i > w_i^*$ and the pairing is alternating ($\epsilon = -1$), in which case $\epsilon_i = -1$.

Lemma 9.16. *Suppose $w_i \neq w_i^*$. The linear transformation $T_i : U_i \rightarrow U'_i$ defined on*

$$(\delta_{w_i w_h})_h \mapsto \left(C_{w_i w_j} = \sum_{h=1}^r \epsilon_h c_{w_j w_h^*} \delta_{w_i w_h} \right)_j$$

is surjective. It is the composition $U_i \rightarrow U \xrightarrow{T} U' \rightarrow U'_i$.

Proof. As I is one-dimensional over $R/\mathfrak{m}_R = k$, it suffices to study the matrix for this linear transformation with respect to a fixed k -basis of I . Fix $w_{h'} \in \mathcal{W}_{\text{FL}, \tau}$. If we take $\delta_{w_i w_h} = 0$ for $w_h \neq w_{h'}$ and $\delta_{w_i w_{h'}} = 1$, the image of $\{\delta_{w_i w_h}\}_h \in U_i$ under T_i has coordinates $C_{w_i w_j} = \epsilon_{w_{h'}} c_{w_j w_{h'}^*}$. Thus the matrix for T_i is

$$\begin{pmatrix} \epsilon_1 c_{w_{i+1} w_1^*} & \epsilon_2 c_{w_{i+1} w_2^*} & \dots & \epsilon_r c_{w_{i+1} w_r^*} \\ \epsilon_1 c_{w_{i+2} w_1^*} & \epsilon_2 c_{w_{i+2} w_2^*} & \dots & \epsilon_r c_{w_{i+2} w_r^*} \\ \dots & \dots & \dots & \dots \\ \epsilon_1 c_{w_r w_1^*} & \epsilon_2 c_{w_r w_2^*} & \dots & \epsilon_r c_{w_r w_r^*} \end{pmatrix}.$$

Multiplying the i th column by ϵ_i , the columns of this matrix are exactly the coordinates of x_{w_j} with respect to the basis $\{n'_w\}_{w \in \mathcal{W}_{\text{FL}, \sigma_\tau}}$ as in Lemma 9.14 except that the first i rows are removed. As the $\{x_w\}$ form a basis, the columns of this matrix span U'_i .

The last statement follows from the definition. \square

Remark 9.17. The statement for $w_i = w_i^*$ is similar. In that case, we must have $\epsilon = 1$, and we have

$$C_{w_i w_i} = 2 \sum_j c_{w_i w_j^*} \delta_{w_i w_j}.$$

Extending the definition of T_i in Lemma 9.16, we again see that the columns of the matrix representing this transformation are truncated versions of the coordinates of x_{w_j} with some signs changed and one coordinate multiplied by 2. The image of a basis under the transformation multiplying one coordinate by 2 is still a basis, so again T_i is surjective.

Lemma 9.18. *The composition $T_{ij} : U_i \rightarrow U \xrightarrow{T} U' \rightarrow U'_j$ is zero whenever $i < j$.*

Informally, this is saying that T is block lower-triangular with diagonal blocks that are surjective.

Proof. The coordinates of U_i are $\delta_{w_i w_h}$. The coordinates of U'_j are $C_{w_j w_h}$ for $j < h$ (or $j \leq h$ if $w_j = w_j^*$). Looking at the formulas for $C_{w_j w_h}$ in the definition of T , they depend only on certain $\delta_{w w'}$ with $w \neq w_i$ as $i < j \leq h$. These are all zero on the image of the inclusion $U_i \rightarrow U$, so the composition is zero. \square

Corollary 9.19. *T is surjective.*

Proof. The composition of $U_i \rightarrow U \rightarrow U' \rightarrow U'_i$ is exactly T_i , hence surjective. For $v \in U'$, by descending induction on i , we will construct $u_i \in U_i$ so that

$$T(u_i + \dots + u_r) - v \in U'_1 \oplus \dots \oplus U'_{i-1}$$

(meaning $T(u_1 + \dots + u_r) = v$ when $i = 1$). For $i = r$, take u_r be a preimage under T_r of the component of v in U'_r . Now suppose we have selected u_{i+1}, \dots, u_r . Pick a preimage $u_i \in U_i$ of the projection of $T(u_{i+1} + \dots + u_r) - v$ to U'_i using the surjectivity of T_i . We know that $T_{ij}(u_i) = 0$ for $j > i$, so

$$T(u_i + \dots + u_r) - v \in U_1 \oplus \dots \oplus U_{i-1}.$$

For $i = 1$, we have $T(u_1 + \dots + u_r) = v$ as desired. \square

Corollary 9.19 lets us choose the $\{\delta_{ih}\}$ so that the compatibility relations (9.1) are satisfied. This defines $\varphi_{M', \tau}^w(n'_w)$, and hence we can extend to a map $\varphi_{M'}^i : M' \rightarrow M'$ compatible with the pairing. We then finish the proof of Proposition 9.8 as follows.

Given the deformation ρ to a coefficient ring R with associated Fontaine-Laffaille module

$$M = \bigoplus_{\tau: W \hookrightarrow \mathcal{O}} M_\tau,$$

and a small surjection $R' \rightarrow R$ whose kernel I is 1-dimensional over the field R/\mathfrak{m}_R , we have constructed a free R' -module M' together with a filtration $\{(M')^i\}$ and maps $\varphi_{M'}^i$ by lifting the M_τ . The filtration and $\{\varphi_{M'}^i\}$ make M' into a Fontaine-Laffaille module. There is an obvious $R' \otimes_{\mathbb{Z}_p} W$ -module structure. The condition $M' = \sum_i \varphi_{M'}^i(M'^i)$ follows from Lemma 9.14. We also constructed a pairing $M' \times M' \rightarrow N$, and the filtration and $\varphi_{M'}^i$ are compatible with it (in the sense of Corollary 9.5) by our choice of $(\delta_{ih})_{ih}$. By Corollary 9.5 and Lemma 8.16, $T_{\text{cris}}(M')$ gives a representation $\rho' : \Gamma_K \rightarrow G(R')$ lifting ρ .

9.3. Tangent Space. The final step in the proof of Theorem 9.2 is to analyze the tangent space of $D_{\bar{\rho}}^{\text{FL}}$. It is a subspace $L_{\bar{\rho}}^{\text{FL}}$ of the tangent space $H^1(\Gamma_K, \text{ad}^0(\bar{\rho}))$ of deformations with fixed similitude character ν . We are mainly interested in its dimension as a vector space over k , and will analyze it by considering deformations ρ of $\bar{\rho}$ to the dual numbers $k[t]/(t^2)$. Recall that $G = \text{GSp}_r$ (with even r) or $G = \text{GO}_r$; let B be a Borel subgroup of G .

Proposition 9.20. *Under the standing assumption that $\bar{\rho}$ is torsion-crystalline with pairwise distinct Fontaine-Laffaille weights for each $\tau : W \hookrightarrow \mathcal{O}$ contained in an interval of length $\frac{p-2}{2}$,*

$$\dim_k L_{\bar{\rho}}^{\text{FL}} - \dim_k H^0(\Gamma_K, \text{ad}^0(\bar{\rho})) = [K : \mathbf{Q}_p](\dim G_k - \dim B_k).$$

Let \bar{V} be the Galois module given by $\bar{\rho}$, and for a lift ρ of $\bar{\rho}$ to $k[t]/(t^2)$ let V be the corresponding Galois module. The submodule tV is naturally isomorphic to \bar{V} , and we have an exact sequence

$$0 \rightarrow tV \rightarrow V \rightarrow \bar{V} \rightarrow 0.$$

Let \bar{M} be the Fontaine-Laffaille module corresponding to $\bar{\rho}$, with pairing $\langle \cdot, \cdot \rangle : \bar{M} \times \bar{M} \rightarrow L_k$. We know \bar{M} is a k -vector space of dimension $r[K : \mathbf{Q}_p]$. Let M be the Fontaine-Laffaille module corresponding to ρ . It is a free $k[t]/(t^2)$ -module, and fits in an exact sequence

$$0 \rightarrow tM \rightarrow M \rightarrow \bar{M} \rightarrow 0$$

of Fontaine-Laffaille modules. The map $\bar{M} \subset M \rightarrow tM$ induced by multiplication by t is an isomorphism of Fontaine-Laffaille modules since it is so on underlying k -vector spaces using the $k[t]/(t^2)$ -freeness of M . As before, we have decompositions

$$M = \bigoplus_{\tau: W \hookrightarrow \mathcal{O}} M_{\tau} \quad \text{and} \quad \bar{M} = \bigoplus_{\tau: W \hookrightarrow \mathcal{O}} \bar{M}_{\tau}$$

from Lemma 8.14.

Using Lemma 9.12, pick a basis $\{v_{\tau,i}\}_{i=1}^r$ of the $k[t]/(t^2)$ -module M_{τ} such that $v_{\tau,i}$ is a basis for a $k[t]/(t^2)$ -complement to $M_{\tau}^{w_i+1}$ in $M_{\tau}^{w_i}$ and such that the pairing $M_{\sigma\tau} \times M_{\sigma\tau} \rightarrow L_{\sigma\tau}$ with respect to the $m_{\tau,i} := \varphi_M^{w_i}(v_{\tau,i})$ is ω_{τ} -times the standard pairing. As 1-units admit square roots, we may assume that $\omega_{\tau} \in k^{\times}$. Note that $\{m_{\tau,i}\} \cup \{tm_{\tau,i}\}$ is a basis for $M_{\sigma\tau}$ as a k -vector space, and $\{m_{\tau,i}\}_{\tau,i}$ is a basis for M as a $k[t]/(t^2)$ module.

Let M_0 be the submodule of M spanned by the $\{v_{\tau,i}\}_{\tau,i}$ as a k -vector space. We have that $tM_0 = tM \simeq \bar{M}$ as vector spaces, and have an obvious decomposition

$$M_0 = \bigoplus_{\tau: W \hookrightarrow \mathcal{O}} M_{\tau,0}.$$

We obtain a pairing on M_0 by restriction and a filtration by intersection: $M_{\tau,0}^i = M^i \cap M_{\tau,0}$.

Lemma 9.21. *We have that $M_{\tau}^i = M_{\tau,0}^i \otimes k[t]/(t^2)$, and hence $M^i = M_0^i \otimes k[t]/(t^2)$.*

Proof. We know that the $k[t]/(t^2)$ -span of v_i is a $k[t]/(t^2)$ -complement to $M_{\tau}^{w_i+1}$ in $M_{\tau}^{w_i}$. Hence $M_{\tau}^{w_i}/M_{\tau}^{w_i+1}$ is isomorphic to the k -span of v_i and tv_i . As the filtration is automatically split (M_{τ}^i is a direct summand of M_{τ} , and hence M_{τ}^i is a direct summand of M_{τ}^{i-1}), this suffices. \square

Observe that the surjection of Fontaine-Laffaille modules $M \rightarrow \bar{M}$ carries M_0 isomorphically onto \bar{M} . Under the isomorphism of k -vector spaces $M_0 \rightarrow \bar{M}$, the pairing on M_0 and the pairing on \bar{M} are identified because by choice of basis the pairing on M_0 is a k^{\times} -multiple of the standard pairing. Furthermore, extending the pairing $M_0 \times M_0 \rightarrow L$ by $k[t]/(t^2)$ -bilinearity recovers the pairing on M . Using $M_0 \simeq \bar{M}$, we can also define $\varphi_{M_0}^i : M_0^i \rightarrow M_0$ to be the lift of $\varphi_{\bar{M}}^i$ to M_0^i . It is compatible with the pairing on M_0 . Note that it is *not* the same as $\varphi_M^i|_{M_0^i}$.

Our goal is to describe the set of strict equivalence classes of deformations M of \bar{M} , so by making these identifications it remains to study ways to lift $\varphi_{\bar{M}}^i$ to a map

$$\varphi_{M_0 \otimes k[t]/(t^2)}^i : M_0^i \otimes k[t]/(t^2) \rightarrow M_0 \otimes k[t]/(t^2).$$

For $n, n' \in M_0^i$ we may write

$$\varphi_M^i(n + tn') = \varphi_{M_0}^i(n) + t(\varphi_{M_0}^i(n') + \delta_i(n))$$

for some σ -semilinear $\delta_i : M_0^i \rightarrow M_0$ which completely determines φ_M^i . It is clear that for $n \in M_0^{i+1}$ we have $\delta_i(n) = 0$ due to the relation $\varphi_{M_0}^i(n) = p\varphi_{M_0}^{i+1}(n) = 0$. Thus, δ_i factors through M_0^i/M_0^{i+1} , and together the δ_i define a σ -semilinear

$$\delta : \mathrm{gr}^\bullet(M_0) \rightarrow M_0.$$

Compatibility with the pairing says exactly that

$$\langle \varphi_M^i(n + tn'), \varphi_M^j(m + tm') \rangle = \varphi_L^{i+j}(\langle n + tn', m + tm' \rangle)$$

for $n, n' \in M_0^i$ and $m, m' \in M_0^j$ and all i and j . Expanding and using the compatibility of the $\varphi_{M_0}^i$ with the pairing, we see that it is necessary and sufficient that

$$(9.2) \quad \langle \delta_i(n), \varphi_{M_0}^j(m) \rangle + \langle \varphi_{M_0}^i(n), \delta_j(m) \rangle = 0$$

for $n \in M_0^i$ and $m \in M_0^j$ and all i and j . As $\overline{M} = \sum_i \varphi_{\overline{M}}^i(\overline{M}^i)$ and we defined $\varphi_{M_0}^i$ to lift $\varphi_{\overline{M}}^i$, it follows that $M_0 = \sum_i \varphi_{M_0}^i(M_0^i)$. Furthermore, we have an isomorphism $\varphi : \mathrm{gr}^\bullet(M_0) \rightarrow M_0$. This allows us to rewrite (9.2) as the requirement that for $m, n \in \mathrm{gr}^\bullet(M_0)$,

$$\langle \delta' \varphi(n), \varphi(m) \rangle + \langle \varphi(n), \delta' \varphi(m) \rangle = 0$$

where δ' is the k -linear composition of φ^{-1} with δ . In other words,

$$\langle \delta' x, y \rangle + \langle x, \delta' y \rangle = 0$$

for all $x, y \in M_0$. Note that δ' is compatible with the filtration, the pairing, and the $k \otimes W$ -module structure. Denote the collection of all such δ' by $\mathrm{End}_{k \otimes W}(M_0, \langle \cdot, \cdot \rangle)$: it is isomorphic to $\mathfrak{sp}_r(k \otimes W)$ or $\mathfrak{so}_r(k \otimes W)$, which have dimension $[K : \mathbf{Q}_p](\dim G_k - 1)$ over k .

Lemma 9.22. *For such a choice of δ' , we obtain a Fontaine-Laffaille module $M \in \mathrm{MF}_{W, \mathrm{tor}}^f$ together with a pairing $M \times M \rightarrow L$ as in Corollary 9.5.*

Proof. This is just bookkeeping. First, observe that $\sum_i \varphi_M^i(M^i)$ is a $k[t]/(t^2)$ -module containing $\varphi_{M_0}^i(M_0^i) = M_0$. Thus it is M . It is immediate that the pairing is compatible with the filtration. We chose δ' so that the pairing is compatible with the φ_M^i . \square

Of course, different δ' may give isomorphic deformations of \overline{M} . Suppose that we are given δ and γ such that the Fontaine-Laffaille modules they create are strictly equivalent as deformations of \overline{M} . We have shown that the underlying module, pairing, and filtration can be identified with the fixed data $M = M_0 \otimes k[t]/(t^2)$, $\langle \cdot, \cdot \rangle \otimes k[t]/(t^2)$, and $M_0^i \otimes k[t]/(t^2)$. The isomorphism reduces to the identity modulo t (by strictness). This means there exists an isomorphism $\alpha : M_0 \rightarrow M_0$ compatible with the pairing, filtration, and module structure such that

$$(1 + t\alpha) (\varphi_{M_0}^i(n) + t(\varphi_{M_0}^i(n') + \delta_i(n))) = \varphi_{M_0}^i(n) + t(\varphi_{M_0}^i(\alpha(n) + n') + \gamma_i(n)).$$

Simplifying, this is the condition that

$$\gamma_i(n) - \delta_i(n) = \alpha(\varphi_{M_0}^i(n)) - \varphi_{M_0}^i(\alpha(n)).$$

In other words, $\delta, \gamma \in \mathrm{End}(M_0, \langle \cdot, \cdot \rangle)$ define the same deformation if and only if $\gamma_i - \delta_i$ is of the form $\alpha \circ \varphi_{M_0}^i - \varphi_{M_0}^i \circ \alpha$ for all i and some $k \otimes W$ -linear $\alpha : M_0 \rightarrow M_0$ that is compatible with the filtration, pairing, and module structures. Under the identification of $\mathrm{End}_{k \otimes W}(M_0, \langle \cdot, \cdot \rangle)$ with the Lie algebra of a symplectic or orthogonal group valued in $k \otimes W$, these are the elements in the Lie algebra of the Borel subgroup B corresponding to the filtration. (The assumption that the Fontaine-Laffaille weights for each τ are pairwise distinct is what makes it a Borel subgroup.) This has dimension $[K : \mathbf{Q}_p](\dim B_k - 1)$ as a k -vector space, since the Borel subgroup of the derived group of G has co-dimension-1 in a Borel subgroup of G .

Finally, we must understand when α and β satisfy

$$\alpha \circ \varphi_{M_0}^i - \varphi_{M_0}^i \circ \alpha = \beta \circ \varphi_{M_0}^i - \varphi_{M_0}^i \circ \beta.$$

This happens exactly when $\alpha - \beta$ commutes with the $\varphi_{M_0}^i$ (as well as being compatible with the filtration, pairing, and module structure). In other words, $\alpha - \beta \in \text{End}_{\text{MF}_W}(M_0, \langle \cdot, \cdot \rangle)$. But under T_{cris} , this is identified with endomorphisms of $\bar{\rho}$ preserving the pairing (not just up to a similitude factor), and in particular has dimension $\dim_k H^0(\Gamma_K, \text{ad}^0(\bar{\rho}))$.

We can express this analysis as the exact sequence

$$0 \rightarrow \text{End}_{\text{MF}_W}(M_0, \langle \cdot, \cdot \rangle) \rightarrow \text{Fil}^0(\text{End}_{k \otimes W}(M_0, \langle \cdot, \cdot \rangle)) \rightarrow \text{End}_{k \otimes W}(M_0, \langle \cdot, \cdot \rangle) \rightarrow D_{\bar{\rho}}^{\text{FL}}(k[t]/(t^2)) \rightarrow 0.$$

We finish the proof of Proposition 9.20 taking dimensions:

$$\begin{aligned} \dim_k L_{\bar{\rho}}^{\text{FL}} - \dim_k H^0(\Gamma_K, \text{ad}^0(\bar{\rho})) &= [K : \mathbf{Q}_p](\dim G_k - 1) - [K : \mathbf{Q}_p](\dim B_k - 1) \\ &= [K : \mathbf{Q}_p](\dim G_k - \dim B_k). \end{aligned}$$

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